Separating two notions of finiteness

Shuwei Wang*

25th October 2023

Abstract

A. R. D. Mathias mentioned in a paper that the axioms of Δ_0-separation and Π_1-foundation suffice to show that two (set-theoretic) characterisations of finiteness, namely "carrying a double well-ordering" and "in bijection with a natural number", coincide. However, it is recently brought to the author's attention that Mathias' proof of the claim is flawed. In this exposition I shall argue that the two aforementioned notions are in fact not equivalent under these axioms, by constructing a counterexample using model-theoretic techniques.

In Mathias' paper [3], one of the systems he considered is ReS, defined to consist of the axioms of extensionality, empty set, pairing, difference, union, Δ_0-separation and Π_1-foundation. Proposition 2.1 in [3] asserts that

Claim 1 (ReS)**.** *If a set X carries a double well-ordering, then it is in bijection with some member of ω.*

The key process in the incorrect proof Mathias provided involves some Π_1 class

$$Z = \{x \in X : \neg\exists f \ f \text{ is an attempt at } x\} \subseteq X,$$

where f is an *attempt* at x if it bijects the initial segment ending at x into an initial segment of ordinals. Mathias argued that if Z is non-empty, it shall have a $\leq_X$-minimal element where $\leq_X$ is a double well-ordering on X, leading to a contradiction. However, Π_1-foundation only ensures that Z has an $\in$-minimal element and in order to get a $\leq_X$-minimal element using the fact that $\leq_X$ is a well-ordering, one needs to invoke Π_1-separation and justify that Z is a set — which is beyond the capabilities of ReS.

In this exposition, we shall show that Claim 1 is in fact false. Namely, our counterexample uses the concept that is commonly known as *rudimentary functions.* The properties of these $V^n \to V$ functions are thoroughly studied in [1], so we will follow the naming convention there and call them *basic functions* instead. We shall define

$$\mathcal{H} = \{\iota^n(n) : n \in \omega\}$$

where $\iota(x) = \{x\}$ and

$$\begin{aligned}\mathcal{R} = &\{\langle \iota^n(n), \iota^m(m)\rangle : n \text{ even}, m \text{ odd}\}\\ &\cup \{\langle \iota^n(n), \iota^m(m)\rangle : n, m \text{ even}, n < m\}\\ &\cup \{\langle \iota^n(n), \iota^m(m)\rangle : n, m \text{ odd}, n > m\}\end{aligned}$$

*School of Mathematics, University of Leeds, Leeds LS2 9JT, UK. E-mail: mmsw@leeds.ac.uk.

so that $\mathcal{R}$ is a strict linear order relation on $\mathcal{H}$.

Let $\mathrm{Bc}(x)$ denote the *basic closure* of x, i.e.

$$\mathrm{Bc}(x) = \{B(\overline{y}) : \overline{y} \in \mathrm{Tc}(x)^n,\ B : V^n \to V \text{ basic function}\},$$

where $\mathrm{Tc}(x)$ denotes the transitive closure of x. We let $\mathcal{U} = \mathrm{Bc}(\{\omega, \mathcal{H}, \mathcal{R}\})$. Then as an immediate corollary of Theorem 1.4.7 and 1.4.8 in [1], we have

Proposition 2. $\mathcal{U} \vDash \mathrm{ReS}$.

Since $\mathcal{U}$ obviously cannot contain a bijection between $\mathcal{H}$ and any member of ω, it suffices now to show

$$\mathcal{U} \vDash \langle \mathcal{H}; \mathcal{R} \rangle \text{ is a double well-ordering.}$$

Observe that $\langle \mathcal{H}; \mathcal{R} \rangle$ actually has order type $\omega \# \omega^*$, where ω^* denotes the reverse ordering of ω. Thus, if $A \in \mathcal{U}$, $A \subseteq \mathcal{H}$ is a set without an $\mathcal{R}$-maximum, then $\{n \in \omega : \iota^n(n) \in A\}$ must be an unbounded set of even numbers. Hence

$$\left\{\iota^{2n}(2n) : n \in \omega\right\} = \{x \in \mathcal{H} : \exists y \in A\, \langle x, y \rangle \in \mathcal{R}\} \in \mathcal{U}$$

by Δ_0-separation. A similar argument holds for the sets without $\mathcal{R}$-minima. Therefore, it suffices to show that

Theorem 3. $\left\{\iota^{2n}(2n) : n \in \omega\right\} \notin \mathcal{U}$.

This shall be the main goal of this exposition. We first show that

Lemma 4. *Suppose that $A \in \mathcal{U}$ and $A \subseteq \mathcal{H}$, then there exists a Δ_0 formula φ such that*

$$A = \{x \in \mathcal{H} : \varphi(x, \mathcal{H}, \mathcal{R})\}.$$

Proof. Observe that sets in $\mathrm{Tc}(\{\omega, \mathcal{H}, \mathcal{R}\}) \setminus \{\omega, \mathcal{H}, \mathcal{R}\}$ are all hereditarily finite. Thus, there must exist a tuple of hereditarily finite sets $\overline{z} \in \mathrm{HF}^{n-3}$ and a basic function $B : V^n \to V$ such that $A = B(\overline{z}, \omega, \mathcal{H}, \mathcal{R})$. Let $I : V^2 \to \{0, 1\}$ denote the basic function

$$I(x, y) = \begin{cases} 1 & \text{if } x \in y, \\ 0 & \text{otherwise,} \end{cases}$$

then $\langle x, \overline{y} \rangle \mapsto I(x, B(\overline{y})) \in \{0, 1\}$ must be the characteristic function of some Δ_0 relation ψ satisfying

$$A = \{x \in \mathcal{H} : \psi(x, \overline{z}, \omega, \mathcal{H}, \mathcal{R})\},$$

by Theorem 1.3.6 in [1].

To eliminate the parameters $\overline{z}$ and ω, we use Theorem 2.1.2 in [1] that the constant function $c_\omega : x \mapsto \omega$ is *substitutable*, i.e. for any Δ_0 relation $\varphi(x, \overline{y})$, there exists a Δ_0 relation $\tilde{\varphi}(x, \overline{y})$ such that

$$\forall x, \overline{y}\, (\varphi(c_\omega(x), \overline{y}) \leftrightarrow \tilde{\varphi}(x, \overline{y})).$$

It is trivial that the constant functions $c_z : x \mapsto z$, where $z \in \mathrm{HF}$, are also substitutable (because they are basic), thus we can find a Δ_0 relation φ such that

$$\forall x, y_1, y_2\, (\psi(x, c_{\overline{z}}(x), c_\omega(x), y_1, y_2) \leftrightarrow \varphi(x, y_1, y_2)).$$

It follows that $A = \{x \in \mathcal{H} : \varphi(x, \mathcal{H}, \mathcal{R})\}$. □

Now, consider the transitive set $\mathcal{M} = \{\iota^m(n) : n, m \in \omega\}$. We shall work in a language $\mathcal{L}^* = \{\in, H, R\}$ where H is a unary relation symbol and R is a binary relation symbol. We show that

Lemma 5. *Suppose that $A = \{x \in \mathcal{H} : \varphi(x, \mathcal{H}, \mathcal{R})\}$ for a Δ_0 formula φ (in the language of set theory $\mathcal{L}_{\text{set}} = \{\in\}$), then there exists a formula φ^* in the language $\mathcal{L}^*$ such that*

$$A = \{x \in \mathcal{H} : \langle \mathcal{M}; \in, \mathcal{H}, \mathcal{R}\rangle \vDash \varphi^*(x)\},$$

where every unbounded quantifier in φ^* is of the form $\forall x\,(Hx \to \cdots)$ or $\exists x\,(Hx \wedge \cdots)$.

Proof. In the language $\mathcal{L}_{\text{set}}$, we introduce new abbreviations

$$\begin{aligned} \forall x R y\ \eta(x,y) &\quad\Rightarrow\quad \forall p \in \mathcal{R}\ \forall x, y \in \mathcal{H}\,(p = \langle x, y\rangle \to \eta(x,y)),\\ \exists x R y\ \eta(x,y) &\quad\Rightarrow\quad \exists p \in \mathcal{R}\ \exists x, y \in \mathcal{H}\,(p = \langle x, y\rangle \wedge \eta(x,y)). \end{aligned}$$

Consider rewriting rules

$$\begin{aligned} \forall p \in \mathcal{R}\ \eta(p) &\quad\Rightarrow\quad \forall x R y\ \eta^*,\\ \exists p \in \mathcal{R}\ \eta(p) &\quad\Rightarrow\quad \exists x R y\ \eta^*,\\ p \in \mathcal{R} &\quad\Rightarrow\quad \exists x R y\ p = \langle x, y\rangle, \end{aligned}$$

where η^* is a Δ_0 relation equivalent to $\eta(\langle x, y\rangle)$ but does not mention the pairing function explicitly, which must exist by Gandy's theory of substitutable functions in [1]. Likewise, $p = \langle x, y\rangle$ stands as an abbreviation for the defining formula of ordered pairs instead of mentioning the pairing function explicitly.

By iterating this rewriting process on $\varphi(x, \mathcal{H}, \mathcal{R})$, we can obtain a formula $\psi(x, \mathcal{H}, \mathcal{R})$ such that every occurrence of $\mathcal{R}$ in ψ is either to the left of the relation symbol $\in$, or part of the bounded quantifier in one of the two abbreviations we just defined. It is easy to show by induction that $\varphi(x, \mathcal{H}, \mathcal{R}) \leftrightarrow \psi(x, \mathcal{H}, \mathcal{R})$.

Finally, we obtain $\varphi^*(x)$ from $\psi(x, \mathcal{H}, \mathcal{R})$ by replacing

$$\begin{aligned} \forall x \in \mathcal{H}\ \eta(x) &\quad\Rightarrow\quad \forall x\,(Hx \to \eta(x)),\\ \exists x \in \mathcal{H}\ \eta(x) &\quad\Rightarrow\quad \exists x\,(Hx \wedge \eta(x)),\\ \forall x R y\ \eta(x,y) &\quad\Rightarrow\quad \forall x, y\,(Hx \wedge Hy \wedge Rxy \to \eta(x,y)),\\ \exists x R y\ \eta(x,y) &\quad\Rightarrow\quad \exists x, y\,(Hx \wedge Hy \wedge Rxy \wedge \eta(x,y)),\\ \mathcal{H} \in x &\quad\Rightarrow\quad \bot,\\ \mathcal{R} \in x &\quad\Rightarrow\quad \bot. \end{aligned}$$

It is again easy to show by induction that

$$\psi(x, \mathcal{H}, \mathcal{R}) \leftrightarrow \langle \mathcal{M}; \in, \mathcal{H}, \mathcal{R}\rangle \vDash \varphi^*(x)$$

for any $x \in \mathcal{M}$ by observing that any such x is hereditarily finite, so $\mathcal{H}, \mathcal{R} \notin x$. ☐

We will simplify the theory of definable subsets of $\mathcal{M}$ by proving a quantifier elimination result. Let $\mathcal{L}^\dagger = \mathcal{L}^* \cup \{0, S, i\}$ be an expanded language with an extra constant symbol and two unary function symbols. Correspondingly, set

$$\mathcal{S}(x) = \begin{cases} \iota^{2n+2}(2n+2) & \text{if } x = \iota^{2n}(2n),\\ \iota^{2n+1}(2n+1) & \text{if } x = \iota^{2n+3}(2n+3),\\ \varnothing & \text{otherwise.} \end{cases}$$

So that $\mathcal{S}$ is the successor function for the linear order $\mathcal{R}$. Let $\mathfrak{S} = \langle \mathcal{M}; \in, \mathcal{H}, \mathcal{R}, \varnothing, \mathcal{S}, \iota \rangle$ denote our standard structure in the language $\mathcal{L}^\dagger$. Observe that we have

Lemma 6. *The first-order theory $T = \mathrm{Th}(\mathfrak{S})$ eliminates "unbounded quantifiers restricted to the domain H", that is, for any formula φ where every unbounded quantifier in φ is of the form $\forall x\,(Hx \to \cdots)$ or $\exists x\,(Hx \wedge \cdots)$, there exists a formula ψ without unbounded quantifiers, such that*

$$T \vDash \forall \overline{x}\,(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))\,.$$

Proof. Let ψ be a formula in $\mathcal{L}^\dagger$ without unbounded quantifiers. It suffices to find another formula $\tilde{\psi}$ without unbounded quantifiers such that

$$T \vDash \forall \overline{x} \left(\exists y\,(Hy \wedge \psi(\overline{x}, y)) \leftrightarrow \tilde{\psi}(\overline{x}) \right).$$

To this end, we can imitate the classical model-theoretic trick in Theorem 3.1.4 in [2]. Denote $\eta(\overline{x}) = \exists y\,(Hy \wedge \psi(\overline{x}, y))$, and let

$$\Gamma = \{\theta(\overline{a}) : \theta \text{ has no unbounded quantifiers, } T \vDash \forall \overline{x}\,(\eta(\overline{x}) \to \theta(\overline{x}))\}\,,$$

then by compactness it suffices to show that $T \cup \Gamma \vDash \eta(\overline{a})$. Suppose otherwise, let $\mathfrak{M}$ be a model of $T \cup \Gamma \cup \{\neg\eta(\overline{a})\}$, and let $T_{\mathfrak{M}}$ be the Δ_0 theory of $\mathfrak{M}$ in the language $\mathcal{L}^\dagger$ with additional constant symbols $\overline{a}$, then $T \cup T_{\mathfrak{M}} \cup \{\eta(\overline{a})\}$ must be satisfiable. Define $\mathfrak{N}$ to be a model of $T \cup T_{\mathfrak{M}} \cup \{\eta(\overline{a})\}$, and we shall derive a contradiction by showing that $\mathfrak{N} \vDash \eta(\overline{a})$ implies $\mathfrak{M} \vDash \eta(\overline{a})$.

We will prove this by analysing the structure of models of T. Firstly, T asserts that all non-singleton sets (and $1 = \{\varnothing\}$) together forms a model of $\mathrm{Th}(\langle \omega; < \rangle)$, that is, a linear order of order type $\omega \,\#\, \mathbb{Z} \cdot \ell$ for some arbitrary linear order ℓ (where the $(\cdot)$ operator denotes the Cartesian product with inverse lexicographic order). "Above" each element x in this class, there must lie a separate sequence of singletons $\{\iota^n(x)\}_{n \in \mathbb{N}^+}$. There can be additional singleton elements, but they must lie in separate sequences of the form $\cdots \in x_{-2} \in x_{-1} \in x_0 \in x_1 \in \cdots$. Finally, the interpretation of H must contain precisely the elements $\{\iota^n(n) : n \in \omega\}$ together with at most one element from each infinite sequence of singletons above, and the relation symbol R must arrange the elements in the interpretation of H into a model of $\mathrm{Th}(\langle \omega \,\#\, \omega^*; < \rangle)$, that is, a linear order of order type $\omega \,\#\, \mathbb{Z} \cdot \ell' \,\#\, \omega^*$ for some arbitrary linear order ℓ', where the elements $\{\iota^n(n) : n \in \omega\}$ occupy the two ends in the same order as given in the standard structure $\langle \mathcal{M}; \in, \mathcal{H}, \mathcal{R} \rangle$ and the singleton elements in the infinite sequences occupy the $\mathbb{Z} \cdot \ell'$ part in the middle.

Let $A^{\mathfrak{M}} \subseteq \mathfrak{M}$ be the smallest transitive substructure containing $\overline{a}^{\mathfrak{M}}$ and also closed under $\left(S^{\mathfrak{M}}\right)^{-1}$ wherever the inverse is defined, that is, $A^{\mathfrak{M}}$ contains any $x \in \mathfrak{M}$ such that for some $a_i^{\mathfrak{M}}$ and $j, k \in \mathbb{Z}$, $\iota^j(x)$ is of finite distance from $\iota^k\left(a_i^{\mathfrak{M}}\right)$ in either of the orderings $\in$ or $R^{\mathfrak{M}}$. Since $T_{\mathfrak{M}}$ contains all atomic formulae that constrain the relative position of pairs a_i, a_j, there is an obvious isomorphism between $A^{\mathfrak{M}}$ and the similarly defined substructure $A^{\mathfrak{N}} \subseteq \mathfrak{N}$. It follows that if $\mathfrak{N} \vDash \psi(\overline{a}, b)$ for some $b \in H^{\mathfrak{N}} \cap A^{\mathfrak{N}}$, then there must be a corresponding $b' \in H^{\mathfrak{M}} \cap A^{\mathfrak{M}}$ such that $\mathfrak{M} \vDash \psi(\overline{a}, b')$.

Suppose otherwise, i.e. $\mathfrak{N} \vDash \psi(\overline{a}, b)$ only for some $b \in H^{\mathfrak{N}} \setminus A^{\mathfrak{N}}$. By compactness, we can then construct a model $\mathfrak{O} \supseteq A^{\mathfrak{N}}$ of T such that $\mathfrak{O} \vDash Hb \wedge \psi(\overline{a}, b)$ for some $b \in \mathfrak{O}$, yet there is a $c \in \mathfrak{O}$ satisfying $c \in H^{\mathfrak{O}} \setminus A^{\mathfrak{N}}$, $Rcx \leftrightarrow Rbx$ for any $x \in H^{\mathfrak{O}} \cap A^{\mathfrak{N}}$ and

$\neg\psi(\overline{a}, c)$ — because given any finite subset of the constraints for c above, we can find some $c^* \in H^{\mathfrak{N}} \cap A^{\mathfrak{N}}$ satisfying them, for which $\mathfrak{N} \vDash \neg\psi(\overline{a}, c^*)$ holds. By our analysis of models of T above, we can easily find an automorphism of $\mathfrak{O}$ that swaps b and c while preserving $A^{\mathfrak{N}}$. Consequently,

$$\mathfrak{O} \vDash \psi(\overline{a}, b) \leftrightarrow \psi(\overline{a}, c).$$

This is a contradiction. Therefore, we must always be in the case above where $\mathfrak{N} \vDash \exists y\,(Hy \wedge \psi(\overline{a}, y))$, and the lemma is proven. □

Lastly, we need to show that

Lemma 7. *Let $\varphi(x)$ be a formula in $\mathcal{L}^\dagger$ with no unbounded quantifiers and exactly one free variable. Then the set*

$$\{x \in \mathcal{H} : \mathfrak{S} \vDash \varphi(x)\}$$

is either finite or cofinite in $\mathcal{H}$.

Proof. Given a formula $\varphi(\overline{x}, \overline{y})$, any $\overline{a} \in \mathcal{M}^n$, any $\overline{k} \in \mathbb{Z}^m$ and any $\overline{s} \in \omega^m$, we show that there exists a large enough $N \in \omega$ such that for any $n > N$,

$$\mathfrak{S} \vDash \varphi\big(v\big(n, \overline{k}, \overline{s}\big), \overline{a}\big) \leftrightarrow \varphi\big(v\big(N, \overline{k}, \overline{s}\big), \overline{a}\big),$$

by induction on the complexity of φ, where

$$v(n, k_i, s_i) = \begin{cases} \iota^{n+2s_i p(n)+k_i}(n + 2s_i p(n)) & \text{if } n \text{ is even,} \\ \iota^{n-2s_i p(n)+k_i}(n - 2s_i p(n)) & \text{if } n \text{ is odd} \end{cases}$$

(with $v(n, k_i, s_i) = \varnothing$ if any computation yields a negative result) and the complexity includes both the number of connectives and the number of function symbols in the formula.

In the base case, observe that any $a \in \mathcal{M}$ is finite, so we can always ensure

$$v(n, k_i, s_i) \notin a \quad \text{and} \quad a \notin v(n, k_i, s_i)$$

for some fixed k_i, s_i when n is large enough; also observe that for any $z \in \mathcal{H}$, the sets $\{x \in \mathcal{H} : \langle x, z\rangle \in \mathcal{R}\}$ and $\{x \in \mathcal{H} : \langle z, x\rangle \in \mathcal{R}\}$ are both either finite or cofinite. The cases for other atomic formulae are similar or trivial. Especially, observe that whether two sets $v(n, k_i, s_i)$ and $v(n, k_j, s_j)$ are related by the relations $\in$ or $\mathcal{R}$ are both determined by the parameters s_i, s_j, k_i, k_j and not affected by the parity of n when n is large enough.

In the inductive case, when φ contains a function symbol whose parameter is simply x_i or y_i, notice that we can simply replace the function term by the result of its invocation and apply the inductive hypothesis. When the parameter is x_i, this is done by inserting a new variable and use either indices $\overline{k}' = \big\langle \overline{k}, k_i \big\rangle, \overline{s}' = \langle \overline{s}, s_i + 1\rangle$ for the function symbol S or $\overline{k}' = \big\langle \overline{k}, k_i + 1 \big\rangle, \overline{s}' = \langle \overline{s}, s_i\rangle$ for the function symbol i.

When φ is of the form $\forall z \in x_i\ \psi(\overline{x}, z, \overline{y})$ or $\exists z \in x_i\ \psi(\overline{x}, z, \overline{y})$, note that $v(n, k_i, s_i) = \{v(n, k_i - 1, s_i)\}$ when n is large enough, so the desired conclusion follows from the inductive hypothesis on the formula ψ and indices $\overline{k}' = \big\langle \overline{k}, k_i - 1 \big\rangle, \overline{s}' = \langle \overline{s}, s_i\rangle$. The rest of the cases only involve finite unions, intersections and complements due to the fact that every $a \in \mathcal{M}$ is finite.

Thus, for the formula $\varphi(x)$ in the lemma, we know by induction that, for some large enough $N \in \omega$,

$$\mathfrak{S} \vDash \varphi(\iota^n(n)) \leftrightarrow \varphi\big(\iota^N(N)\big),$$

for any $n > N$. The lemma follows immediately. □

The lemmata 4 through 7 imply that for any set $A \in \mathcal{U}$ such that $A \subseteq \mathcal{H}$, A must be either finite or cofinite in $\mathcal{H}$. Therefore Theorem 3 holds, that is, the set $\big\{\iota^{2n}(2n) : n \in \omega\big\}$, which is neither finite nor cofinite in $\mathcal{H}$, cannot be in $\mathcal{U}$. The model $\mathcal{U}$ thinks that $\mathcal{H}$ carries a double well-ordering $\mathcal{R}$, but is not in bijection with any member of ω. This contradicts Claim 1.

References

[1] Robin Oliver Gandy, *Set-theoretic functions for elementary syntax*, Axiomatic set theory, part 2 (Thomas J. Jech, ed.), Proceedings of Symposia in Pure Mathematics, vol. 13, American Mathematical Society, 1974, pp. 103–126.

[2] David E. Marker, *Model theory: An introduction*, Graduate Texts in Mathematics, Springer New York, NY, 2002.

[3] Adrian Richard David Mathias, *Weak systems of Gandy, Jensen and Devlin*, Set theory: Centre de Recerca Matemàtica Barcelona, 2003–2004 (Joan Bagaria and Stevo Todorcevic, eds.), Trends in Mathematics, Birkhäuser Basel, 2006, pp. 149–224.