The global well-ordering on Weaver's third-order conceptual mathematics

Shuwei Wang

University of Leeds

12th September 2024

Weaver's conceptual mathematics

Axioms for the global well-ordering

The predicativity problem

Mathematic conceptualism

Nik Weaver, *Mathematical conceptualism* (2005), `arXiv:math/0509246 [math.LO]`:

> *[We] have now reached a point where we have a fairly clear idea of just what portion of the Cantorian universe is relevant to mainstream mathematics ... this region is, with remarkable accuracy, precisely the portion a conceptualist would recognize as legitimate.*

Nik Weaver, *Axiomatizing mathematical conceptualism in third order arithmetic* (2009), `arXiv:0905.1675 [math.HO]`:

> *[Mathematical conceptualism] is a refinement of the predicativist philosophy of Poincaré and Russell. The basic idea is that we accept as legitimate only those structures that can be constructed, but we allow constructions of transfinite length.*

The axioms of CM

Decidability for all atomic formulae:

$$n = m \vee \neg n = m,$$
$$n \in X \vee \neg n \in X,$$
$$X \in \mathbf{X} \vee \neg X \in \mathbf{X};$$

Numerical omniscience:

$$\forall n\,(\varphi(n) \vee \psi(n)) \rightarrow (\forall n\,\varphi(n) \vee \exists n\,\psi(n))\,;$$

Comprehension axioms for all decidable formulae:

$$\forall n\,(\varphi(n) \vee \neg\varphi(n)) \rightarrow \exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))\,,$$
$$\forall X\,(\varphi(X) \vee \neg\varphi(X)) \rightarrow \exists \mathbf{X}\,\forall X\,(X \in \mathbf{X} \leftrightarrow \varphi(X))\,;$$

The axioms of CM

The standard *number axioms* PA^-;

Full induction:

$$\varphi(0) \wedge \forall n\, (\varphi(n) \to \varphi(n+1)) \to \forall n\, \varphi(n);$$

Dependent choice:

$$\begin{aligned}&\forall n\, \forall X\, \exists Y\, \varphi(n, X, Y)\\ &\quad \to \forall X\, \exists Z\, \big((Z)_0 = X \wedge \forall n\, \varphi\big(n, (Z)_n, (Z)_{n+1}\big)\big)\,;\end{aligned}$$

Extensionality for second-order variables:

$$X = Y \to (X \in \mathbf{X} \leftrightarrow X \in \mathbf{Y})\,.$$

Mathematics in CM

Usual definitions for $N, Z, Q, \mathbf{R}$;

Theorem (CM)

$\mathbf{R}$ is a sequentially complete ordered field. Every sequentially complete ordered field is isomorphic to $\mathbf{R}$.

Theorem (CM)

Let $\mathbf{K}$ be a separable subset of $\mathbf{R}$. Then TFAE:

(i) $\mathbf{K}$ is closed and bounded;

(ii) $\mathbf{K}$ is compact;

(iii) $\mathbf{K}$ is bounded and contains the limits of all Cauchy sequences;

(iv) every sequence in $\mathbf{K}$ has a subsequence which converges to a limit in $\mathbf{K}$.

Mathematics in CM

We can also reason about

- metric spaces (preferably separable);
- topological spaces (preferably second countable);
- Banach spaces (preferably separable).

Theorem (Hahn–Banach, in CM)

Let $\mathbf{E}$ be a separable Banach space, let $\mathbf{E}_0$ be a separable closed subspace, and let $\mathbf{f}_0 : \mathbf{E}_0 \to \mathbf{R}$ be a bounded linear functional on $\mathbf{E}_0$. Then $\mathbf{f}_0$ extends to a bounded linear functional $\mathbf{f}$ on $\mathbf{E}$ with $\|\mathbf{f}\| = \|\mathbf{f}_0\|$.

The proof-theoretic strength of CM

We can interpret CM in Σ^1_1-AC, using Σ^1_1-*(partial) functions*.

Notation

Let $\exists X\, \pi(e, m, a, b, X)$ be a universal Σ^1_1-formula, i.e. for any Σ^1_1-formula $\varphi(m, a, b)$ there exists $e \in \mathbb{N}$ such that

$$\forall m \forall a \forall b\, (\varphi(m, a, b) \leftrightarrow \exists X\, \pi(e, m, a, b, X))\,.$$

We denote $\{e\}(a) = b \quad :\Leftrightarrow \quad \exists X\, \pi(e_0, e_1, a, b, X)$, and write

$$\begin{aligned} \mathrm{dom}(e) &:= \{a : \exists! b\, \{e\}(a) = b\}\,, \\ \mathrm{ran}(e, X) &:= \{b : \exists a \in X\, \{e\}(a) = b\}\,, \\ [X, Y] &:= \{e : X \subseteq \mathrm{dom}(e) \wedge \mathrm{ran}(e, X) \subseteq Y\}\,. \end{aligned}$$

The proof-theoretic strength of CM

Let second-order variables range over $S_2 = [\mathbb{N}, \{0,1\}]$, third-order variables range over $S_3 = [S_2, \{0,1\}]$ (up to extensional equality).

$$
\begin{aligned}
d \Vdash t = s \quad &:\Leftrightarrow \quad t = s \text{ for any arithmetic terms } t \text{ and } s,\\
d \Vdash t \in e \quad &:\Leftrightarrow \quad \{e\}(t) = 1 \text{ for any arithmetic terms } t,\\
d \Vdash e \in f \quad &:\Leftrightarrow \quad \{f\}(e) = 1,\\
d \Vdash \varphi \wedge \psi \quad &:\Leftrightarrow \quad d_0 \Vdash \varphi \wedge d_1 \Vdash \psi,\\
d \Vdash \varphi \vee \psi \quad &:\Leftrightarrow \quad (d_0 = 0 \wedge d_1 \Vdash \varphi) \vee (d_0 = 1 \wedge d_1 \Vdash \psi),\\
d \Vdash \neg\varphi \quad &:\Leftrightarrow \quad \forall e \, \neg e \Vdash \varphi,\\
d \Vdash \varphi \to \psi \quad &:\Leftrightarrow \quad \forall e \, (e \Vdash \varphi \to e \in \mathrm{dom}(d) \wedge \{d\}(e) \Vdash \psi),\\
d \Vdash \forall x \, \varphi(x) \quad &:\Leftrightarrow \quad d \in [\mathbb{N}, \mathbb{N}] \wedge \forall n \, \{d\}(n) \Vdash \varphi(n),\\
d \Vdash \exists x \, \varphi(x) \quad &:\Leftrightarrow \quad d_1 \Vdash \varphi(d_0),\\
d \Vdash \forall X \, \varphi(X) \quad &:\Leftrightarrow \quad d \in [S_2, \mathbb{N}] \wedge \forall n \in S_2 \, \{d\}(n) \Vdash \varphi(n),\\
d \Vdash \exists X \, \varphi(X) \quad &:\Leftrightarrow \quad d_0 \in S_2 \wedge d_1 \Vdash \varphi(d_0),\\
d \Vdash \forall \mathbf{X} \, \varphi(\mathbf{X}) \quad &:\Leftrightarrow \quad d \in [S_3, \mathbb{N}] \wedge \forall n \in S_3 \, \{d\}(n) \Vdash \varphi(n),\\
d \Vdash \exists \mathbf{X} \, \varphi(\mathbf{X}) \quad &:\Leftrightarrow \quad d_0 \in S_3 \wedge d_1 \Vdash \varphi(d_0).
\end{aligned}
$$

The proof-theoretic strength of CM

Theorem (Σ^1_1-AC)

For every axiom φ of CM, $\exists d\ d \Vdash \varphi$.

Observe that Σ^1_1-AC^i is an immediate subtheory of CM, and we have $\left|\Sigma^1_1\text{-}\mathrm{AC}^i\right| = \left|\Sigma^1_1\text{-}\mathrm{AC}\right|$ due to [Aczel, 1977].

Corollary

$|\mathrm{CM}| = \left|\Sigma^1_1\text{-}\mathrm{AC}\right| = \varphi_{\varepsilon_0}(0)$.

Axioms for the global well-ordering

Relation $\prec$ is a linear ordering between second-order objects;

Transfinite induction:

$$\forall X\,(\forall Y\,(Y \prec X \to \varphi(Y)) \to \varphi(X)) \to \forall X\,\varphi(X);$$

Countability of initial segments:

$$\forall X\,\exists Z\,\forall Y\,(Y \prec X \to \exists n\, Y = (Z)_n)\,.$$

Note. The transfinite induction axiom is obviously impredicative.

Zorn's lemma and mathematics with GWO

Theorem ($\mathrm{CM} + \mathrm{GWO}$)

Fix a third-order set $\mathbf{A}$ *and let* $\varphi(X)$ *be a decidable formula (possibly with other parameters) on the countable subsets of* $\mathbf{A}$, *i.e. we have* $\forall X \subseteq \mathbf{A}\, (\varphi(X) \vee \neg\varphi(X))$. *If* φ *as a predicate is downward closed and also closed under unions of countable chains, then there exists some* $\mathbf{X} \subseteq \mathbf{A}$ *satisfying* $\forall X \subseteq \mathbf{X}\ \varphi(X)$, *and is maximal among such subsets in the sense that*

$$\forall Y\, (Y \notin \mathbf{X} \rightarrow \exists Z \subseteq \mathbf{X}\ \neg\varphi(Z \cup \{Y\}))\,.$$

Zorn's lemma and mathematics with GWO

Corollary

Let **V** *be an* **F***-vector space, such that the formula*

$$\varphi(k, X) = \text{“}(X)_0, \ldots, (X)_k \text{ are not all } 0 \text{ and generate } 0.\text{”}$$

is decidable, then **V** *has a basis* $\mathbf{X} \subseteq \mathbf{V}$ *such that every vector* $V \in \mathbf{V}$ *is spanned by finitely many elements in* **X**.

Examples of vector spaces with a basis:

- vector spaces over a countable field (e.g. **R** over Q);
- normed real/complex vector spaces, e.g. a Banach space;
- any inner product spaces.

The proof-theoretic strength of GWO

Let $\mathcal{O}$ be the class of recursive well-orderings. Let $\mathcal{H}$ be the set of pairs $\langle a, e\rangle$ such that $a \in \mathcal{O}$ and $\Phi_e^{H_a}$ is a total function.

Fact (ATR_0)

Every $d \in S_2$ denotes the same function as some $h \in \mathcal{H}$, and vice versa.

Theorem (ATR_0)

There exist Σ^1_1-partial functions $\mathrm{hyp} \in [S_2, \mathcal{H}]$ and $\mathrm{inv} \in [S_2, \mathcal{H}]$ that represent the conversions above.

The proof-theoretic strength of GWO

Let BI be ACA_0 plus the axiom scheme

$$\mathsf{WO}(X,<_X) \to \forall x \in X\,(\forall y <_X x\; \varphi(y) \to \varphi(x)) \to \forall x \in X\, \varphi(x).$$

We can interpret $\prec$ as the natural well-ordering on the hyperarithmetic indices and have

Theorem (BI)

For every axiom φ of $\mathrm{CM} + \mathrm{GWO}$, $\exists d\; d \Vdash \varphi$.

For the lower bound, one can interpret ID_1^i in $\mathrm{CM} + \mathrm{GWO}$, and we have $\left|\mathrm{ID}_1^i\right| = |\mathrm{BI}|$ due to [Buchholz & Pohlers, 1978].

Corollary

$|\mathrm{CM} + \mathrm{GWO}| = |\mathrm{BI}| = \theta_{\varepsilon_{\Omega+1}}(0).$

Weaver's problem with impredicativity

The axiom of transfinite induction in GWO is impredicative:

$$\forall X\,(\forall Y\,(Y \prec X \to \varphi(Y)) \to \varphi(X)) \to \forall X\,\varphi(X).$$

Weaver did not like this. In [Weaver, 2009]:

> *This should only be asserted for formulas that do not contain $\prec$, for reasons having to do with the circularity involved in making sense of a relation that is well-ordered with respect to properties that are defined in terms of that relation.*

But this does not solve the problem, since we can use a third-order parameter

$$\mathbf{E} = \{\langle X, Y\rangle : X \prec Y\}\,.$$

ATR and totally realisable formulae

Observe that every axiom in $\mathrm{CM} + \mathrm{GWO}$ other than unrestricted transfinite induction is realisable in ATR. We also have:

Fact (Well-ordering principle)

Over RCA_0, ATR_0 is equivalent to $\forall X\,(\mathsf{WO}(X) \to \mathsf{WO}(\varphi_X))$.

This means for some sufficiently simple Σ^1_1-function $f : \mathcal{O} \to \mathcal{O}$, ATR_0 implies that $\mathcal{O}$ is still closed under f iterated along any well-ordering $a \in \mathcal{O}$.

ATR and totally realisable formulae

We say that φ is *totally realisable* if there exists a Δ^1_1-formula $\chi_\varphi(e)$ and Σ^1_1-(partial) functions f, g such that

$$d \Vdash \varphi \to d \in \mathrm{dom}(f) \wedge \{f\}(d) \in S_2 \wedge \chi_\varphi(\{f\}(d)),$$
$$e \in S_2 \wedge \chi_\varphi(e) \to e \in \mathrm{dom}(g) \wedge \{g\}(e) \Vdash \varphi.$$

Proposition (Σ^1_1-AC)

The class of totally realisable formulae includes all arithmetic formulae, is closed under quantifiers $\exists Y \prec X$ and $\forall Y \prec X$, and can contain positive occurrences of $\exists X$.

ATR and totally realisable formulae

Theorem (ATR)

Let $\varphi(X)$ *be a totally realisable formula and* $f \in [S_2, S_2]$ *such that*

$$\forall e \in S_2 \left(\chi_{\forall Y \prec X\ \varphi(Y)}(e) \to \chi_{\varphi(X)}(\{f\}(e))\right),$$

and $\mathcal{O}$ *is closed under* $b \mapsto \sup_{\langle b,e\rangle \in \mathcal{H}}\{\mathrm{hyp}\}(\{f\}(\{inv\}(\langle b, e\rangle)))_0$ *iterated along any well-ordering* $a \in \mathcal{O}$*, then* $\exists d\ d \Vdash \forall X\ \varphi(X)$*.*

For example, this suffices to show that the Hamel basis of $\mathbf{R}$ over Q exists.

Question

(i) Can we do this in a theory weaker than ATR?

(ii) How can we formulate the condition for f syntactically?

Thank you!

[1] Peter Aczel, *The strength of Martin-Löf's intuitionistic type theory with one universe*, Proceedings of the symposiums on mathematical logic in Oulo 1974 and in Helsinki 1975 (Seppo Miettinen and Jouko Väänänen, eds.), 1977, pp. 1–32.

[2] Wilfried Buchholz and Wolfram Pohlers, *Provable wellorderings of formal theories for transfinitely iterated inductive definitions*, The Journal of Symbolic Logic **43** (1978), no. 1, 118–125.

[3] Michael Rathjen, *Well-ordering principles in proof theory and reverse mathematics*, Axiomatic thinking II (Fernando Ferreira, Reinhard Kahle, and Giovanni Sommaruga, eds.), Springer Cham, 2022, pp. 89–127.

[4] Stephen George Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, 2009.

[5] Nik Weaver, *Mathematical conceptualism* (2005), available at `arXiv:math/0509246[math.LO]`.

[6] Nik Weaver, *Axiomatizing mathematical conceptualism in third order arithmetic* (2009), available at `arXiv:0905.1675[math.HO]`.