Realisability semantics and choice principles for Weaver's third-order conceptual mathematics

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Mathematical conceptualism in (Weaver, 2005)

Classical set theory presents us with a picture of an incredibly vast universe. [. . .] Yet virtually all important objects in mainstream mathematics are either countable or separable. This is not because the uncountable/nonseparable case has not yet been sufficiently studied but rather because on mainstream questions it tends to be either pathological or undecidable.

Mathematical conceptualism in (Weaver, 2005)

Any domain in which set-theoretic reasoning is to take place must be in some sense constructed. [. . . And] in order for a construction to be considered valid it need not be physically realizable, but it must be conceptually definite, meaning that we must be able to form a completely clear mental picture of how the construction would proceed.

Surveyable and definite collections

As described in (Weaver, 2011):

- *a concept is* surveyable *if it is possible, in principle, to exhaustively survey all of the individuals which fall under it; whereas*
- *a concept is* definite *if any individual either does or does not fall under it.*

For a more detailed analysis of Weaver's philosophy, see upcoming: Michael Rathjen and Shuwei Wang, *Recent developments on predicative foundations*, Pillars of Enduring Strength: Learning from Hermann Weyl (Laura Crosilla, Øystein Linnebo and Michael Rathjen, eds.).

CM in (Weaver, 2009a)

Third-order intuitionistic arithmetic with:

- induction: for any $\varphi(n)$,

$$\varphi(0) \wedge \forall n\,(\varphi(n) \to \varphi(n+1)) \to \forall n\,\varphi(n);$$

- recursion/dependent choice: for any $\varphi(n, X, Y)$,

$$\forall n \forall X\, \exists Y\, \varphi(n, X, Y) \to \forall X\, \exists Z\, \left((Z)_0 = X \wedge \forall n\, \varphi\left(n, (Z)_n, (Z)_{n+1}\right)\right);$$

- limited principle of omniscience:

$$\forall n\,(\varphi(n) \vee \psi(n)) \to \forall n\,\varphi(n) \vee \exists n\,\psi(n);$$

- decidable comprehension:

$$\forall n\,(\varphi(n) \vee \neg\varphi(n)) \to \exists X\, \forall n\,(n \in X \leftrightarrow \varphi(n)),$$
$$\forall X\,(\varphi(X) \vee \neg\varphi(X)) \to \exists \mathbf{X}\, \forall X\,(X \in \mathbf{X} \leftrightarrow \varphi(X));$$

A glimpse of mathematics in CM

Theorem ((Weaver, 2009a), 3.10)

$\mathbb{R}$ is a sequentially complete ordered field. Every sequentially complete ordered field is isomorphic to $\mathbb{R}$.

Theorem (Baire category theorem, (Weaver, 2009a), 3.41)

The intersection of any countable family of open dense subsets of a separable (Cauchy) complete metric space is dense.

Theorem (Hahn–Banach theorem, (Weaver, 2009a), 3.72)

Let $\mathbf{E}$ be a separable Banach space, let $\mathbf{E}_0$ be a separable closed subspace, and let $\mathbf{f}_0 : \mathbf{E}_0 \to \mathbb{R}$ be a bounded linear functional on $\mathbf{E}_0$. Then $\mathbf{f}_0$ extends to a bounded linear functional $\mathbf{f}$ on $\mathbf{E}$ with $\|\mathbf{f}\| = \|\mathbf{f}_0\|$.

Extending CM

In order to admit choice principles on uncountable sets (e.g. sets of reals), we need to extend CM. In (Weaver, 2009a, section 2.3), Weaver proposed:

- $\prec$ is a global linear ordering on second-order objects;
- transfinite induction:

$$\forall X\, (\forall Y\, (Y \prec X \to \varphi(Y)) \to \varphi(X)) \to \forall X\, \varphi(X);$$

- countable initial segments:

$$\forall X\, \exists Z\, \forall Y\, (Y \prec X \to \exists n\, Y = (Z)_n)\,.$$

We shall henceforth denote these as the *global well-ordering axioms* (GWO).

Zorn's lemma

Proposition ($\mathrm{CM} + \mathrm{GWO}$, (W., 2025), 3.2)

Fix a third-order set $\mathbf{A}$ *and let* $\varphi(X)$ *be a decidable formula (possibly with other parameters) on the countable subsets of* $\mathbf{A}$. *If* φ *as a predicate is*

- *downward closed; and also*
- *closed under unions of countable chains,*

then there exists some $\mathbf{X} \subseteq \mathbf{A}$ *satisfying* $\forall X \subseteq \mathbf{X}\ \varphi(X)$, *and is maximal among such subsets in the sense that*

$$\forall Y\,(Y \notin \mathbf{X} \to \exists Z \subseteq \mathbf{X}\ \neg\varphi(Z \cup \{Y\}))\,.$$

Some corollaries of Zorn's lemma

- $\mathbb{R}$ has a basis as a $\mathbb{Q}$-vector space;
- uncountable inner product spaces and $\mathbb{R}$-normed spaces also have bases;
- Hahn–Banach theorem for a non-separable Banach space $\mathbf{E}$;
- etc.

Partial combinatory algebra

A partial combinatory algebra (PCA) with arithmetic is a set $\mathcal{A} \supseteq \mathbb{N}$ with a partial binary operation, such that there exist the following distinguished objects in $\mathcal{A}$:

- $k \cdot a \cdot b = a$,
- $s \cdot a \cdot b \cdot c = a \cdot c \cdot (b \cdot c)$,
- $\mathrm{succ} \cdot n = n + 1$ for $n \in \mathbb{N}$,
- $\mathrm{pred} \cdot n = n - 1$ for $n \in \mathbb{N}^+$,
- $d \cdot a \cdot b \cdot c \cdot d = \begin{cases} a & \text{if } c = d, \\ b & \text{if } c \neq d, \end{cases}$
- and pairing and unpairing functions p, p_0, p_1.

Arithmetic decidability

To capture the Limited Principle of Omniscience, we need a (higher) notion of computability that decides arithmetical quantifiers. That is, a PCA $\mathcal{A}$ such that

> *There is a function $e \in \mathcal{A}$ such that, for any total $f : \mathbb{N} \to \mathbb{N}$ in the PCA,*
> $$e \cdot f = f'.$$

Σ^1_1-functions

Working in classical second-order arithmetic, Kleene's normal form theorem gives the following universal Σ^1_1-formula (with all free variables indicated) where π is (universal) Π^0_1:

$$\exists X\, \pi(e, m, a, b, X).$$

We define

$$\begin{aligned} \{e\}(a) = b &\quad\Leftrightarrow\quad \exists X\, \pi(e_0, e_1, a, b, X), \\ \{e\}(a)\downarrow &\quad\Leftrightarrow\quad \exists b!\, \{e\}(a) = b. \end{aligned}$$

We can write

$$\begin{aligned} \mathrm{dom}(e) &\coloneqq \{a : \{e\}(a)\downarrow\}\,, \\ \mathrm{ran}(e, X) &\coloneqq \{b : \exists a \in X\, \{e\}(a) = b\}\,, \\ [X, Y] &\coloneqq \{e : X \subseteq \mathrm{dom}(e) \wedge \mathrm{ran}(e, X) \subseteq Y\}\,. \end{aligned}$$

Σ^1_1-functions

Proposition (S-m-n theorem, (W., 2025), 2.3)

Given any definable class A of natural numbers and a Σ^1_1-formula $\varphi(m, a, b)$ such that, for any $a \in A$, there exists a unique $b \in \mathbb{N}$ satisfying $\varphi(a_0, a_1, b)$, then there exists e such that for any $a \in A$, we have $a_0 \in \mathrm{dom}(e)$, $a_1 \in \mathrm{dom}(\{e\}(a_0))$ and

$$\varphi(a_0, a_1, \{\{e\}(a_0)\}(a_1)).$$

Corollary

The set of natural numbers $\mathbb{N}$ form a partial combinatory algebra with arithmetic under the application operation $\{-\}(-)$.

Σ^1_1-axiom of choice

Over the base theory of arithmetic comprehension, ACA_0, Σ^1_1-AC_0 add the following axiom:

$$\forall x \, \exists Y \, \varphi(x, Y) \to \exists Z \, \forall x \, \varphi(x, Z_x)$$

for any arithmetic formula φ.

In Σ^1_1-AC_0 we can show that the pointclass of (equivalently) Σ^1_1-formulae is closed under arithmetic quantifiers. It follows that the PCA of Σ^1_1-functions decides arithmetic quantifiers.

Realisability conditions

Let second-order variables range over $S_2 = [\mathbb{N}, \{0,1\}]$, and third-order variables range over $S_3 = [S_2, \{0,1\}]$. We define:

$$
\begin{aligned}
d \Vdash t = s &\Leftrightarrow t = s \text{ for any arithmetic terms } t \text{ and } s,\\
d \Vdash t \in_1 e &\Leftrightarrow \{e\}(t) = 1 \text{ for any arithmetic terms } t,\\
d \Vdash e \in_2 f &\Leftrightarrow \{f\}(e) = 1,\\
d \Vdash \varphi \wedge \psi &\Leftrightarrow d_0 \Vdash \varphi \wedge d_1 \Vdash \psi,\\
d \Vdash \varphi \vee \psi &\Leftrightarrow (d_0 = 0 \wedge d_1 \Vdash \varphi) \vee (d_0 = 1 \wedge d_1 \Vdash \psi),\\
d \Vdash \neg\varphi &\Leftrightarrow \forall e \, \neg e \Vdash \varphi,\\
d \Vdash \varphi \to \psi &\Leftrightarrow \forall e \, (e \Vdash \varphi \to e \in \mathrm{dom}(d) \wedge \{d\}(e) \Vdash \psi),\\
d \Vdash \forall x \, \varphi(x) &\Leftrightarrow d \in [\mathbb{N}, \mathbb{N}] \wedge \forall n \, \{d\}(n) \Vdash \varphi(n),\\
d \Vdash \exists x \, \varphi(x) &\Leftrightarrow d_1 \Vdash \varphi(d_0),\\
d \Vdash \forall X \, \varphi(X) &\Leftrightarrow d \in [S_2, \mathbb{N}] \wedge \forall n \in S_2 \, \{d\}(n) \Vdash \varphi(n),\\
d \Vdash \exists X \, \varphi(X) &\Leftrightarrow d_0 \in S_2 \wedge d_1 \Vdash \varphi(d_0),\\
d \Vdash \forall \mathbf{X} \, \varphi(\mathbf{X}) &\Leftrightarrow d \in [S_3, \mathbb{N}] \wedge \forall n \in S_3 \, \{d\}(n) \Vdash \varphi(n),\\
d \Vdash \exists \mathbf{X} \, \varphi(\mathbf{X}) &\Leftrightarrow d_0 \in S_3 \wedge d_1 \Vdash \varphi(d_0).
\end{aligned}
$$

Realisability model

Fact

Let $\mathcal{A}$ be any PCA, if the sentence φ is a theorem in intuitionistic logic, then there exists $d \in \mathcal{A}$ such that $d \Vdash \varphi$.

Theorem (Σ^1_1-AC (with full induction), (W., 2025), 2.10)

Over the previously defined specific PCA, whenever

$$\mathrm{CM} \vdash \varphi(x, \ldots, X, \ldots, \mathbf{X}, \ldots),$$

there exists $d \in \mathbb{N}$ such that $\{d\}(\vec{p}) \Vdash \varphi(\vec{p})$ for any valid parameter assignment $\vec{p}$.

Ordinal analysis of CM

Let $\Sigma^1_1\text{-}\mathrm{AC}^i$ denote the intuitionistic fragment of $\Sigma^1_1\text{-}\mathrm{AC}$, then it is due to (Aczel, 1977) that

$$\Sigma^1_1\text{-}\mathrm{AC}^i \equiv_{\mathsf{Con}} \Sigma^1_1\text{-}\mathrm{AC}.$$

Observe that $\Sigma^1_1\text{-}\mathrm{AC}^i$ is trivially a subtheory of CM. Thus

Corollary

The proof-theoretic strength of CM *is*

$$|\mathrm{CM}| = \left|\Sigma^1_1\text{-}\mathrm{AC}\right| = \varphi_{\varepsilon_0}(0).$$

Hyperarithmetic sets

Given a universal Π^0_1-formula $\pi(e, m, X)$, we define the *Turing jump*

$$\mathsf{TJ}(X) = \{\langle e, m\rangle : \pi(e, m, X)\}.$$

Let α denote a recursive ordinal, then the *iterated Turing jump* $\mathsf{TJ}^\alpha(X)$ is a set Y such that

$$\forall\beta \leq \alpha\ Y_\beta = \mathsf{TJ}(Y_{<\beta}).$$

A set X is *hyperarithmetic* if there exists a recursive ordinal α such that X is Turing reducible to $\mathsf{TJ}^\alpha(\varnothing)$.

Theorem (Suslin–Kleene theorem, ATR_0)

A set is Δ^1_1-definable if and only if it is hyperarithmetic.

Well-ordering of $S_2 = [\mathbb{N}, \{0, 1\}]$

Theorem (ATR_0, (W., 2025), 4.3)

There exists hyp *such that*

- *for each* $e \in S_2$, $\{\mathrm{hyp}\}(e)$ *computes a pair* $\langle \alpha, r \rangle$ *such that*

$$\{e\} = \Phi_r^{\mathsf{TJ}^\alpha(\varnothing)};$$

- *for* $d, e \in S_2$, $\{\mathrm{hyp}\}(d) = \{\mathrm{hyp}\}(e)$ *if and only if* $\forall n \, \{d\}(n) = \{e\}(n)$.

Proposition (ACA_0)

There exists inv *such that, for any recursive well-ordering* α,

$$\{\{\{\mathrm{inv}\}(\langle \alpha, r \rangle)\}\} = \Phi_r^{\mathsf{TJ}^\alpha(\varnothing)}.$$

Well-ordering of $S_2 = [\mathbb{N}, \{0,1\}]$

Fact (ATR_0)

Recursive well-orderings are Σ^1_1-comparable. Specifically, for any recursive ordinal α, the set of recursive well-orderings $\leq \alpha$ is arithmetic in $\mathsf{TJ}^\alpha(\varnothing)$.

Let cmp denote the following computation: for input d, e, we compute $\{\mathrm{hyp}\}(d) = \langle \alpha, r\rangle$ and $\{\mathrm{hyp}\}(e) = \langle \beta, s\rangle$ (if they converge). We set $\{\{\mathrm{cmp}\}(d)\}(e) = 1$ if all of the following:

- $\beta \leq \alpha$,
- either $\alpha \not\leq \beta$, or $\alpha \leq \beta$ and $\{\mathrm{hyp}\}(e) < \{\mathrm{hyp}\}(d)$ as natural numbers,
- $\Phi_s^{\mathsf{TJ}^\beta(\varnothing)}$ is total $\mathbb{N} \to \{0,1\}$.

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Theorem (ATR_0, (W., 2025))

- cmp *computes a well-ordering on S_2 (up to extensionality).*
- *For $d \in S_2$, $\{\mathrm{hyp}\}(d) = \langle \alpha, r \rangle$, the initial segment of this well-ordering before d is arithmetic in $\mathsf{TJ}^\alpha(\varnothing)$.*

Realisability of GWO

BI denotes ACA_0 with the additional axiom

$$\forall X\,(\mathrm{WO}(X) \to \forall x \in X\,(\forall y <_X x\; \varphi(y) \to \varphi(x)) \to \forall x \in X\,\varphi(x))\,.$$

Theorem (BI, (W., 2025), §4.3-4)

Interpret the global well-ordering $\prec$ as cmp, then whenever

$$\mathrm{CM} + \mathrm{GWO} \vdash \varphi(x, \ldots, X, \ldots, \mathbf{X}, \ldots),$$

there exists $d \in \mathbb{N}$ such that $\{d\}(\vec{p}) \Vdash \varphi(\vec{p})$ for any valid parameter assignment $\vec{p}$.

Ordinal analysis of $\mathrm{CM} + \mathrm{GWO}$

Theorem ((W., 2025))

For any strictly positive first-order formula $\varphi(X, x)$, there exists $\theta_\varphi(x)$ such that

$$\mathrm{CM} + \mathrm{GWO} \vdash \forall x \left(\varphi(\theta_\varphi, x) \to \theta_\varphi(x)\right),$$

and also for any formula $\eta(x)$

$$\mathrm{CM} + \mathrm{GWO} \vdash \forall x \left(\varphi(\eta, x) \to \eta(x)\right) \to \forall x \left(\theta_\varphi(x) \to \eta(x)\right).$$

In other words, the first-order intuitionistic theory of inductive definitions ID_1^i is interpretable in $\mathrm{CM} + \mathrm{GWO}$.

Ordinal analysis of $\mathrm{CM} + \mathrm{GWO}$

The proof-theoretic ordinal of ID_1^i is computed in (Buchholz and Pohlers, 1978) as $\left|\mathrm{ID}_1^i\right| = \theta_{\varepsilon_{\Omega+1}}(0) = |\mathrm{BI}|$. Thus

Corollary

The proof-theoretic strength of $\mathrm{CM} + \mathrm{GWO}$ *is the Bachmann–Howard ordinal*

$$|\mathrm{CM} + \mathrm{GWO}| = |\mathrm{BI}| = \theta_{\varepsilon_{\Omega+1}}(0).$$

Weaver worries about impredicative transfinite induction

[transfinite induction] should only be asserted for formulas that do not contain $\prec$*, for reasons having to do with the circularity involved in making sense of a relation that is well-ordered with respect to properties that are defined in terms of that relation.*

Weaver worries about impredicative transfinite induction

> *[transfinite induction] should only be asserted for formulas that do not contain $\prec$, for reasons having to do with the circularity involved in making sense of a relation that is well-ordered with respect to properties that are defined in terms of that relation.*

. . . Only that this literal restriction does not work, since formulae can contain higher-order parameters, which in turn can involve $\prec$ in non-trivial ways:

$$\mathbf{W} \quad := \quad \{\langle X, Y\rangle \mid X \prec Y\}\,.$$

Fragments of $\mathrm{CM} + \mathrm{GWO}$

- Any decidable formula $\varphi(X)$ is equivalent to $X \in \mathbf{X}$ for some parameter $\mathbf{X}$; we can let GWO_0 include the single induction axiom

$$\forall X\, (\forall Y \prec X\ Y \in \mathbf{X} \rightarrow X \in \mathbf{X}) \rightarrow \forall X\, X \in \mathbf{X}.$$

- Likewise, let GWO_Σ include the single induction axiom

$$\forall X\, (\forall Y \prec X\ \exists Z\, \langle Y, Z \rangle \in \mathbf{X} \rightarrow \exists Z\, \langle X, Z \rangle \in \mathbf{X}) \\ \rightarrow \forall X\, \exists Z\, \langle X, Z \rangle \in \mathbf{X}.$$

Both are nice classes of formulae, closed under arithmetic quantifiers $\forall n$, $\exists n$ and bounded quantifiers $\forall Y \prec X$, $\exists Y \prec X$.

Fragments of $\mathrm{CM} + \mathrm{GWO}$

Theorem

The fragment $\mathrm{CM} + \mathrm{GWO}_0$ *is interpreted in* ATR *(with full induction);*
The fragment $\mathrm{CM} + \mathrm{GWO}_\Sigma$ *is interpreted in* Π^1_2-TI*.*

We have

$$|\mathrm{ATR}| = \Gamma_{\varepsilon_0}, \qquad \left|\Pi^1_2\text{-}\mathrm{TI}\right| = \theta_{\Omega^{\varepsilon_0}}(0)$$

(the latter is due to (Rathjen and Weiermann, 1993)). It is yet *open* whether these upper bounds for fragments of $\mathrm{CM} + \mathrm{GWO}$ are tight.

Mathematics in weak fragments

Proposition

$\mathrm{CM} + \mathrm{GWO}_\Sigma$ *suffices for Zorn's lemma in (W., 2025).*

(We need a slightly modified proof than in the original paper.)

Mathematics in weak fragments

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For specific results, we may have weaker bounds, e.g.

Proposition

ATR *interprets* $\mathrm{CM} + \mathrm{GWO}_0$ *together with:*

$$\mathbb{R} \textit{ has a basis as a } \mathbb{Q}\textit{-vector space.}$$

(But there is no good axiomatisation of this theory.)

Weaver's predicativity

Since Weaver's conceptualism claims that we have a predicative (mental) grasp of any countable procedure, he agrees with the following criticism of the Feferman–Schütte limit in (Howard, 1996):

Let $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$ be an increasing sequence of recursive ordinals with limit Γ_0. Suppose that for each n, the predicativist has a valid formal justification that γ_n is well-ordered. Then he should be able to *reflect* on the predicative validity of his formal systems and infer that "for each n, γ_n is indeed well-ordered", thus justify the well-orderedness of Γ_0 and beyond.

Weaver's strong "predicative" systems

In (Weaver, 2009b; 2022), he tried to produce such strong systems by iterating truth predicates and reflection principles. He had

- $\mathrm{Tarski}^{\omega}_{\Gamma_0}(\mathrm{PA})$ proves transfinite induction up to any ordinal less than Γ_0 for all formulae in its language;
- $\mathrm{Tarski}^{\omega}_{\kappa^\kappa}(\mathrm{PA})$ proves transfinite induction up to any ordinal less than $\theta_{\Omega^2}(0)$ for all formulae in its language;
- $\mathrm{Tarski}^{\omega}_{\lambda^{\lambda^\omega}}(\mathrm{PA})$ proves transfinite induction up to any ordinal less than $\theta_{\Omega^\omega}(0)$ for all formulae in its language;

. . . and claims that the process can continue to at least reach the large Veblen ordinal.

Thank you!

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