Analysing Gödel's L in Realisability Models of CZF

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Constructive Zermelo–Fraenkel set theory CZF

Extensionality + Pairing + Union + Strong Infinity + Set Induction + Δ_0-Separation + Strong Collection + Subset Collection

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Set Induction: $\forall x\,(\forall y \in x\ \varphi(y) \to \varphi(x)) \to \forall x\ \varphi(x)$

Strong Collection: $\forall x \in a\ \exists y\ \varphi(x, y) \to$
$\exists b\,(\forall x \in a\ \exists y \in b\ \varphi(x, y) \wedge \forall y \in b\ \exists x \in a\ \varphi(x, y))$

Subset Collection: $\exists c\ \forall u\,(\forall x \in a\ \exists y \in b\ \varphi(x, y, u) \to$
$\exists d \in c\,(\forall x \in a\ \exists y \in d\ \varphi(x, y, u) \wedge \forall y \in d\ \exists x \in a\ \varphi(x, y, u)))$

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$\exists d \in c\, (\forall x \in a\, \exists y \in d\, \varphi(x, y, u) \wedge \forall y \in d\, \exists x \in a\, \varphi(x, y, u)))$

Exponentiation:
$\forall x\, \forall y$ the set y^x of all functions from x to y exists

Proposition

Powerset $\Rightarrow$ *Subset Collection* $\Rightarrow$ *Exponentiation.*

We denote $\mathrm{CZF}(\mathcal{P}) = \mathrm{CZF} +$ Powerset

Ordinals and L

An *ordinal* is a transitive set of transitive sets. For ordinals α, we construct the usual hierarchy

$$L_\alpha = \bigcup_{\beta \in \alpha} \mathrm{def}(L_\beta)$$

where $\mathrm{def}(L_\beta)$ is the collection of all first-order definable sets in $\langle L_\beta; \in \rangle$ with parameters. Then $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$.

The intuitionistic L was first treated by Robert Lubarsky [5]. Other ways to define L are still intuitionistically equivalent, such as iterating finitely many fundamental operations, as verified recently by Matthews & Rathjen [6].

Intuitionistic ordinals (and L)

Proposition (ZF)

If $\alpha \subseteq \beta$ are ordinals, then either $\alpha = \beta$ or $\alpha \in \beta$. Especially, it follows that Ord is linearly ordered.

The proof of this starts with "either $\beta \subseteq \alpha$, or there exists $\gamma \in \beta$ such that $\gamma \notin \alpha \ldots$", which is not intuitionistically valid!

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The proof of this starts with "either $\beta \subseteq \alpha$, or there exists $\gamma \in \beta$ such that $\gamma \notin \alpha \ldots$", which is not intuitionistically valid! Likewise, the following corollary only works in classical logic:

Corollary (ZF)

If α is an ordinal, then $\alpha = L_\alpha \cap \mathrm{Ord} \in L_{\alpha+1}$.

It remains *open* whether any intuitionistic set theories suffice to prove $\mathrm{Ord} \subseteq L$!

Partial combinatory algebras

A *partial combinatory algebra* (PCA) is a set $\mathcal{A}$ with a partial application operation $\mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A}$, with two distinguished combinators:

- for any $a, b \in \mathcal{A}$, $\mathbf{k}ab\downarrow$ and $\mathbf{k}ab = a$;
- for any $a, b, c \in \mathcal{A}$, $\mathbf{s}ab\downarrow$ and $\mathbf{s}abc \simeq (ac)(bc)$.

Here, we say that a (formal) application term t *converges*, denoted $t\downarrow$ if all application operations involved are defined, and we write $t \simeq s$ if both converges to the same value, or if both diverges.

For example, $\mathcal{A} = \mathbb{N}$ where ab evaluates to the result of running the a^{th} Turing machine on input b is a PCA.

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PCAs give one some generalised notion of computation. For example, we have the following basic properties:

- When t is a (formal) term containing some variable symbol x, then $\lambda x.t$ is also a term in $\mathcal{A}$.
- We have the usual fixed-point combinators in PCAs, so we can define functions by recursion.

Additional structures on PCAs

A *PCA over the natural numbers* is some $\mathcal{A} \supseteq \mathbb{N}$ with:

- $\mathbf{s}_N, \mathbf{p}_N \in \mathcal{A}$ such that for any $n \in \mathbb{N}$, $\mathbf{s}_N n\downarrow = n + 1$ and $\mathbf{p}_N n\downarrow = \max\{n-1, 0\}$;
- definition by cases $\mathbf{d} \in \mathcal{A}$, such that for any terms a, b and $c_1, c_2 \in \mathbb{N}$,

$$\mathbf{d}abc_1c_2 \simeq \begin{cases} a & \text{if } c_1 = c_2, \\ b & \text{otherwise.} \end{cases}$$

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We also often identify distinguished pairing functions $\mathbf{p}, \mathbf{p}_0, \mathbf{p}_1 \in \mathcal{A}$ in a PCA such that

$$\mathbf{p}_0(\mathbf{p}ab) \simeq a, \qquad \mathbf{p}_1(\mathbf{p}ab) \simeq b.$$

Kleene realisability $\mathrm{CZF} \hookrightarrow \mathrm{CZF}$ (Rathjen 2006)

Fix a PCA $\mathcal{A}$ (over the natural numbers), the class of names $V(\mathcal{A}) = \bigcup_{\alpha \in \mathrm{Ord}} V(\mathcal{A})_\alpha$ is given by

$$V(\mathcal{A})_\alpha = \bigcup_{\beta \in \alpha} \mathcal{P}\Big(\mathcal{A} \times V(\mathcal{A})_\beta\Big).$$

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Realisability conditions:

$$\begin{aligned} e \Vdash a \in b \quad &\Leftrightarrow \quad \exists c \left(\langle \mathbf{p}_0 e \downarrow, c \rangle \in b \wedge \mathbf{p}_1 e \Vdash a = c \right), \\ e \Vdash a = b \quad &\Leftrightarrow \quad \forall f, d \left((\langle f, d \rangle \in a \rightarrow \mathbf{p}_0 ef \Vdash d \in b) \right. \\ &\qquad \left. \wedge \left(\langle f, d \rangle \in b \rightarrow \mathbf{p}_1 ef \Vdash d \in a \right) \right). \end{aligned}$$

Proposition

There is a fixed $\mathbf{i} \in \mathcal{A}$ *such that* $\mathbf{i} \Vdash a = a$ *for all* $a \in V(\mathcal{A})$.

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Realisability conditions (continued):

$$
\begin{aligned}
e \Vdash \varphi \wedge \psi \quad &\Leftrightarrow \quad \mathbf{p}_0 e \Vdash \varphi \wedge \mathbf{p}_1 e \Vdash \psi, \\
e \Vdash \varphi \vee \psi \quad &\Leftrightarrow \quad (\mathbf{p}_0 e\downarrow = 0 \wedge \mathbf{p}_1 e \Vdash \varphi) \\
&\qquad \vee (\mathbf{p}_0 e\downarrow = 1 \wedge \mathbf{p}_1 e \Vdash \psi), \\
e \Vdash \neg\varphi \quad &\Leftrightarrow \quad \forall f \in \mathcal{A}\ f \nVdash \varphi, \\
e \Vdash \varphi \to \psi \quad &\Leftrightarrow \quad \forall f \in \mathcal{A}\,(f \Vdash \varphi \to ef \Vdash \psi), \\
e \Vdash \forall x \in a\ \varphi(x) \quad &\Leftrightarrow \quad \forall \langle f, c\rangle \in a\ ef \Vdash \varphi(c), \\
e \Vdash \exists x \in a\ \varphi(x) \quad &\Leftrightarrow \quad \exists c\,(\langle \mathbf{p}_0 e\downarrow, c\rangle \in a \wedge \mathbf{p}_1 e \Vdash \varphi(c)), \\
e \Vdash \forall x\ \varphi(x) \quad &\Leftrightarrow \quad \forall c \in V(\mathcal{A})\ e \Vdash \varphi(c), \\
e \Vdash \exists x\ \varphi(x) \quad &\Leftrightarrow \quad \exists c \in V(\mathcal{A})\ e \Vdash \varphi(c).
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Kleene realisability $\mathrm{CZF} \hookrightarrow \mathrm{CZF}$ (Rathjen 2006)

Theorem (Rathjen, 2006)

CZF *proves that for every theorem* φ *of* CZF, *there is a realiser* $e \in \mathcal{A}$ *such that* $e \Vdash \varphi$.

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Proposition (W.)

There is a realiser $e \in \mathcal{A}$ *such that*

$$e \Vdash \exists \alpha \in \mathrm{Ord}\ \alpha \subsetneq L_\alpha \cap \mathrm{Ord}.$$

"Exotic" ordinals

Here are the usual constructions for ω in $V(\mathcal{A})$:

$$\overline{n} = \{\langle m, \overline{m}\rangle : m \in n\},$$
$$\overline{\omega} = \{\langle n, \overline{n}\rangle : n \in \omega\},$$

then some $e \Vdash \overline{\omega}$ is the smallest inductive set.

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then some $e \Vdash \overline{\omega}$ is the smallest inductive set.

If we consider $\overline{2}' = \{\langle 1, \overline{0}\rangle, \langle 0, \overline{1}\rangle\}$, then some $f \Vdash \overline{2} = \overline{2}'$.

However, any realiser

$$g \not\Vdash \left\{\langle 1, \overline{2}\rangle, \left\langle 1, \overline{2}'\right\rangle\right\} = \{\langle 1, \overline{2}\rangle\},$$

so the left-hand side is actually a *proper* superset!

"Exotic" ordinals

Here, we will look at

$$\overline{3}^* = \left\{ \langle 0, \overline{0} \rangle, \langle 0, \overline{1} \rangle, \langle 1, \overline{2} \rangle, \left\langle 1, \overline{2}' \right\rangle \right\}.$$

We shall sketch a proof that $L_{\overline{3}^*} \cap \mathrm{Ord} \subseteq \overline{3}^*$ is *not* realised!

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We shall sketch a proof that $L_{\overline{3}^*} \cap \mathrm{Ord} \subseteq \overline{3}^*$ is *not* realised!

The general idea is that a same realiser realises $\overline{0} \in \overline{2}$ and $\overline{1} \in \overline{2}'$. From this, also a same realiser realises

$$\overline{0} = L_{\overline{0}} \cap \mathrm{Ord} \in L_{\overline{2}} \quad \text{and} \quad \overline{1} = L_{\overline{1}} \cap \mathrm{Ord} \in L_{\overline{2}'}.$$

Consequently, we have a same realiser for the successors

$$\overline{0}^+ \in \mathrm{def}(L_{\overline{2}}) \subseteq L_{\overline{3}^*} \quad \text{and} \quad \overline{1}^+ \in \mathrm{def}\left(L_{\overline{2}'}\right) \subseteq L_{\overline{3}^*}.$$

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However, suppose that $f \Vdash L_{\overline{3}^*} \cap \mathrm{Ord} \subseteq \overline{3}^*$, while $e \Vdash$ both $\overline{0}^+, \overline{1}^+ \in L_{\overline{3}} \cap \mathrm{Ord}$, then

$$0 = \mathbf{p}_0(f(\mathbf{p}_0 e)) = 1,$$

a contradiction.

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In fact, we can realise

$$\begin{aligned} \overline{3}^* = \{x \in L_{\overline{3}^*} \cap \mathrm{Ord} : (x \subseteq 1 \wedge (\neg\neg 0 \in x \rightarrow 0 \in x)) \vee \\ (\neg\neg x = 2 \wedge \forall y \in x\, (y = 0 \rightarrow 1 \in x) \wedge \exists y, z \in x\, (\neg y = z))\}. \end{aligned}$$

Not an inner model!

A more important recent result proved through a realisability model is

Theorem (Matthews & Rathjen, 2024)

$\mathrm{CZF} \nvdash L \vDash \mathrm{CZF}$.

More specifically, it is shown that even $\mathrm{CZF}(\mathcal{P})$ does not prove that L satisfies the axiom of Exponentiation, a consequence of Subset Collection. Namely, it is shown that

Proposition

$\mathrm{CZF}(\mathcal{P}) \nvdash L \vDash$ *the set of all functions from* ω *to* ω *exists.*

$E_\wp$-recursive functions

We use the definition of a PCA consisting of (class) functions acting on all sets, as given in Rathjen [8]. There we have ω as a constant and additional distinguished combinators:

$$\boldsymbol{\pi} xy \simeq \{x, y\}, \qquad \boldsymbol{\nu} x \simeq \bigcup x,$$

$$\boldsymbol{\gamma} xy \simeq x \cap \bigcap y, \qquad \boldsymbol{\rho} xy \simeq \{xu : u \in y\},$$

$$\mathbf{i}_1 xyz \simeq \{u \in x : y \in z\},$$
$$\mathbf{i}_2 xyz \simeq \{u \in x : u \in y \to u \in z\},$$
$$\mathbf{i}_3 xyz \simeq \{u \in x : u \in y \to z \in u\},$$

$$\wp x \simeq \mathcal{P}(x).$$

Weakened-realisability-with-truth of $\mathrm{CZF}(\mathcal{P})$

The names for this realisability model are just arbitrary sets themselves.

Weakened realisability: we are allowed to produce a (non-empty) set of realisers without actually computing a specific inhabitant.

$$
\begin{aligned}
a \Vdash \varphi \vee \psi \quad &\Leftrightarrow \quad a \neq \varnothing \wedge \forall e \in a \,((\mathbf{p}_0 e\!\downarrow = 0 \wedge \mathbf{p}_1 e \Vdash \varphi) \\
&\qquad\qquad \vee (\mathbf{p}_0 e\!\downarrow = 1 \wedge \mathbf{p}_1 e \Vdash \psi))\,, \\
a \Vdash \exists x \in b\ \varphi(x) \quad &\Leftrightarrow \quad a \neq \varnothing \wedge \forall e \in a \,(\mathbf{p}_0 e\!\downarrow \in b \wedge \mathbf{p}_1 e \Vdash \varphi(\mathbf{p}_0 e))\,, \\
a \Vdash \exists x\ \varphi(x) \quad &\Leftrightarrow \quad a \neq \varnothing \wedge \forall e \in a\ \mathbf{p}_1 e \Vdash \varphi(\mathbf{p}_0 e\!\downarrow).
\end{aligned}
$$

Weakened-realisability-with-truth of $\mathrm{CZF}(\mathcal{P})$

The names for this realisability model are just arbitrary sets themselves.

Realisability *with truth*: any realised formula must simultaneously have a computational realiser AND hold in the meta-theory.

$$\begin{aligned} a \Vdash b \in c &\quad\Leftrightarrow\quad b \in c, \\ a \Vdash \neg\varphi &\quad\Leftrightarrow\quad \neg\varphi \wedge \forall e\; e \nVdash \varphi, \\ a \Vdash \varphi \to \psi &\quad\Leftrightarrow\quad (\varphi \to \psi) \wedge \forall f\,(f \Vdash \varphi \to af \Vdash \psi)\,. \end{aligned}$$

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Proposition (Rathjen, 2012)

For any formula φ, $\mathrm{CZF}(\mathcal{P}) \vdash (\exists a\; a \Vdash_{\mathfrak{wt}} \varphi) \to \varphi$.

Computational content

Theorem (Rathjen, 2012)

For any formula $\varphi(x_1, \ldots, x_n)$ (with all free variables listed), if $\mathrm{CZF}(\mathcal{P}) \vdash \varphi$, then one can effectively construct the index of an $E_\wp$-recursive function f such that

$$\mathrm{CZF}(\mathcal{P}) \vdash \forall a_1, \ldots, a_n \; f a_1 \cdots a_n \Vdash_{\mathfrak{wt}} \varphi(a_1, \ldots, a_n).$$

Proof of $\mathrm{CZF}(\mathcal{P}) \nvdash L \vDash$ Exponentiation

Suppose that

$$\mathrm{CZF}(\mathcal{P}) \vdash \exists \alpha \in \mathrm{Ord}\ \exists x \in L_\alpha\ \forall f : \omega \to \omega\, (f \in L \to f \in x)\,.$$

We use the previous theorems to convert into realisability and back into truth, which means we can find an $E_\wp$-recursive term t that computes to a set of ordinals α satisfying the condition above.

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Now, one key result in the 2012 paper, Rathjen [8], (proved using a variant of this realisability model) is that

Proposition

$\mathrm{CZF}(\mathcal{P})$ *is* $\Pi_2^{\mathcal{P}}$*-conservative over* $\mathrm{IKP}(\mathcal{P})$.

Proof of $\mathrm{CZF}(\mathcal{P}) \nvdash L \vDash$ Exponentiation

Using this conservativity, $\mathrm{IKP}(\mathcal{P})$ already proves that the $E_\wp$-recursive term t evaluates to a set. By Cook & Rathjen's relativised ordinal analysis of $\mathrm{IKP}(\mathcal{P})$ [3], this set additionally lies in V_σ for some recursive ordinal $\sigma <$ BH. In other words,

$$\mathrm{CZF}(\mathcal{P}) \vdash \exists \alpha \in V_\sigma \cap \mathrm{Ord}\, \forall f : \omega \to \omega \, (f \in L_\sigma \to f \in L_\alpha)\,.$$

This sentence is $\Sigma_1^{\mathcal{P}}$, so by conservativity again, it is also provable in $\mathrm{IKP}(\mathcal{P})$ and thus $\mathrm{KP}(\mathcal{P})$. So α is a gap ordinal, but classically, the smallest gap ordinal is much larger than BH, as a result by Leeds & Putnam [4]. A contradiction.

Relativised ordinal analysis

The key step in the preceding proof is the relativised ordinal analysis (first used on the classical theory $\mathrm{KP}(\mathcal{P})$ in Rathjen [10]), which essentially implies that $\mathrm{IKP}(\mathcal{P})$ (and $\mathrm{KP}(\mathcal{P})$) cannot prove the existence of ordinals beyond BH.

In [11], this is applied to show that

Theorem (Rathjen, 2020)

$\mathrm{KP}(\mathcal{P}) + V = L$ *is much stronger than* $\mathrm{KP}(\mathcal{P})$.

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Question

Is there a similar proof that $\mathrm{CZF} + V = L$ is much stronger than CZF?

The $V = L$ model

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The answer is *no*.

Theorem (W.)

$\mathrm{CZF} + V = L$ is equi-consistent with CZF.

The $V = L$ model

Question

Is there a similar proof that $\mathrm{CZF} + V = L$ is much stronger than CZF?

The answer is *no*.

Theorem (W.)

$\mathrm{CZF} + V = L$ *is equi-consistent with* CZF.

We shall sketch an interpretation

$$\mathrm{CZF} + V = L \hookrightarrow \mathrm{ML_1V}^X \hookrightarrow \mathrm{BI} \equiv_{\mathrm{Con}} \mathrm{CZF}.$$

The type theory $\mathrm{ML_1V}^X$

We have the following types:

1. finite types $\overline{n}$ for each $n \in \mathbb{N}$,
2. the type $\overline{\mathbb{N}}$ of natural numbers,
3. an arbitrary type $\overline{X}$, given by a set $X \subseteq \mathbb{N}$ in the interpretation $\mathrm{ML_1V}^X \hookrightarrow \mathrm{BI}$,
4. dependent Σ and Π types,
5. one universe U, closed under the type constructions above,
6. a single W-type denoted V, with the following constructor:

$$\frac{\Gamma \vdash A : U, f : A \to V}{\Gamma \vdash \mathrm{sup}(A, f) : V}$$

Interpretations

The interpretation $\mathrm{CZF} \hookrightarrow \mathrm{ML_1V}$ introduced by Aczel [1] is essentially a realisability model: the names are terms of type V; the notion of computation is given by corresponding function types.

The interpretation $\mathrm{ML_1V} \hookrightarrow \mathrm{BI}$ (which, combined with the above, gives an formal realisability model $\mathrm{CZF} \hookrightarrow \mathrm{BI}$ over the PCA of Turing machines) is then set-up in Rathjen [9].

Subcountability

A crucial feature of the interpretation $\mathrm{ML_1V} \hookrightarrow \mathrm{BI}$ is that every type A is interpreted by a subset of $\mathbb{N}$. This means that the realisability model realises the following set-theoretic axiom:

Subcountability: $\forall x \; \exists y \subseteq \omega \; \exists f : y \to x$ surjection.

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Corollary (CZF + Subcountability)

If every $x \subseteq \omega$ lies in L, then $V = L$.

Proof.

W.l.o.g. consider some transitive $x \in V$. Then $(x, \in) \cong (U, E)$ where $U \subseteq \mathbb{N}$, $E \subseteq \mathbb{N} \times \mathbb{N}$. Now, $U, E \in L$ by assumption, so we can reconstruct x. □

Setting-up for realising $\mathcal{P}(\omega) \subseteq L$

In [5], Lubarsky shows that $\mathrm{CZF} \vdash \forall n \in \omega \ n = L_n \cap \mathrm{Ord}$. It follows:

Lemma

If $\alpha \in \mathcal{P}(\omega) \cap \mathrm{Ord}$, then

$$\alpha = \bigcup_{n \in \alpha} n^+ = \bigcup_{n \in \alpha} \mathrm{def}(L_n) \cap \mathrm{Ord} = L_\alpha \cap \mathrm{Ord}.$$

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Fix $\alpha_0 \in \mathcal{P}(\omega) \cap \mathrm{Ord}$, we can extract $f_0 : \omega \to \mathcal{P}(\omega) \cap \mathrm{Ord}$ given by

$$f_0(i) = \{n \in \omega : \forall k \leq n \; \pi(i, k) \in \alpha_0\} \in L,$$

where $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is some (recursive) pairing bijection.

Incomparable ordinals

We say that $f : \omega \to \mathcal{P}(\omega) \cap \mathrm{Ord}$ is *pairwise incomparable* iff (intuitively) for any $i \neq j \in \omega$, $f(i)$ and $f(j)$ are not subsets of each other. Formally, we want

$$\forall i, j \in \omega \, (f(i) \subseteq f(j) \to i = j) \, .$$

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Proposition

If aforementioned $f_0 \in L$ is pairwise incomparable, then $\mathcal{P}(\omega) \subseteq L$.

Proof.

For any $x \subseteq \omega$, we take $\sigma = \bigcup_{n \in x} \mathrm{def}\left(L_{f_0(n)}\right) \cap \mathrm{Ord} \in L$ and verify

$$x = \{n \in \omega : L_\eta \vDash f_0(n) \in \sigma\} \in L$$

for some large enough $\eta \in \mathrm{Ord}$. □

A priority argument

A ordinal $\alpha \subseteq \omega$ in the realisability model can be (roughly) given by the name $\sup(A, f)$ where A is a type (i.e. a subset of $\mathbb{N}$ in the meta-theory) and $f : A \to \mathbb{N}$ is a recursive bijection. $\omega \not\subseteq \alpha$ is realised iff the inverse of f is not recursive.

More generally, names $\alpha = \sup(A, f)$ and $\beta = \sup(B, g)$ are both not subsets of each other iff both $g^{-1} \circ f$ and $f^{-1} \circ g$ are not recursive.

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Thus, to get the f_0 we need, we want the distinguished type $\overline{X}$ in our $\mathrm{ML_1V}^X$ to interpret some $X \subseteq \mathbb{N}$ satisfying

> $\mathcal{R}_{i,j,f}$: *there exists $m \in \mathbb{N}$ such that if we input the first $\pi(i, m)$ elements of X to the Turing machine Φ_f, it does not compute the first $\pi(j, m)$ elements of X*

for all $i, j, f \in \mathbb{N}$.

A priority argument

> $\mathcal{R}_{i,j,f}$: *there exists* $m \in \mathbb{N}$ *such that if we input the first* $\pi(i, m)$ *elements of* X *to the Turing machine* Φ_f*, it does not compute the first* $\pi(j, m)$ *elements of* X

But this is possible so long as for any $i, j \in \mathbb{N}$, there are arbitrarily large numbers m such that $\pi(i, m) < \pi(j, m)$. Then we just use arithmetic recursion to construct the set X.

We just need to pick an appropriate pairing function. For example,

$$\pi(a, b) = \begin{cases} \max\{a, b\} \cdot (\max\{a, b\} + 1) - a + b & \text{if } \max\{a, b\} \text{ is even,} \\ \max\{a, b\} \cdot (\max\{a, b\} + 1) + a - b & \text{otherwise;} \end{cases}$$

Finally,

Combining all these constructions, we have a realisability model

$$\mathrm{CZF} + \mathsf{Subcountability} + V = L \hookrightarrow \mathrm{BI} \equiv_{\mathrm{Con}} \mathrm{CZF}.$$

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Open Question

What about $\mathrm{CZF}(\mathcal{P})$? Is $\mathrm{CZF}(\mathcal{P}) + V = L$ equi-consistent with $\mathrm{CZF}(\mathcal{P})$ or stronger?

Open Question

It is even harder to construct non-classical models of $V \neq L$. Ultimately, can we violate $\mathrm{Ord} \subseteq L$?

Thank you!

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