An ordinal analysis of CM and its extensions

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21st January 2025

Abstract

In [10], Nik Weaver proposed a novel intuitionistic formal theory of third-order arithmetic as a formalisation of his philosophical position known as *mathematical conceptualism*. In this paper, we will construct a realisability model from the partial combinatory algebra of Σ_1^1-definable partial functions and use it to provide an ordinal analysis of this formal theory. Additionally, we will examine possible extensions to this system by adding well-ordering axioms, which are briefly mentioned but never thoroughly studied in Weaver's work. We aim to use the realisability arguments to discuss how much such extensions constitute an increase from the original theory's proof-theoretic strength.

Contents

1   Introduction

Nik Weaver first introduced his philosophy of *mathematical conceptualism* in [9], which was essentially rejecting the full Cantorian universe of sets as a foundational ontological commitment and instead advocating a restricted language involving only countably and predicatively constructed mathematical objects — for example, Weaver has no problem reasoning about individual real numbers, but views the complete totality $\mathcal{P}(\mathbb{N})$ as a dangerous object to postulate.

In his early papers [8, 9], this philosophical motivation is only reflected by a mild restriction of set theory to the weak fragment J_2, the first two levels in Jensen's hierarchy in the fine structure theory, but Weaver later adopted a more radical foundational viewpoint that quantifiers over infinitary objects (e.g. the individual reals) can be manipulated only in an intuitionistic manner. This led to a shift from using a set-theoretic foundation to one in higher order arithmetic, which Weaver formalised in [10]. The system consists of the following axioms over third-order intuitionistic logic:

 (1) the number axioms, essentially the axioms of PA excluding induction;

 (2) the unrestricted induction scheme

$$\varphi(0) \wedge \forall n\, (\varphi(n) \to \varphi(n+1)) \to \forall n\, \varphi(n);$$

 for any third-order formula $\varphi(n)$;

 (3) the recursion scheme, also known as dependent choice,

$$\forall n\, \forall X\, \exists Y\, \varphi(n, X, Y) \to \forall X\, \exists Z\, \big((Z)_0 = X \wedge \forall n\, \varphi\big(n, (Z)_n, (Z)_{n+1}\big)\big),$$

 for any third-order formula $\varphi(n, X, Y)$, where uppercase X, Y, Z denote second-order variables, and $(Z)_n$ denotes the set $\{m : \langle n, m \rangle \in Z\}$ (which exists by the comprehension axioms below);

 (4) the decidable comprehension axioms

$$\forall n\, (\varphi(n) \vee \neg\varphi(n)) \to \exists X\, \forall n\, (n \in X \leftrightarrow \varphi(n)),$$
$$\forall X\, (\varphi(X) \vee \neg\varphi(X)) \to \exists \mathbf{X}\, \forall X\, (X \in \mathbf{X} \leftrightarrow \varphi(X)),$$

 where X denotes a second-order variable and boldface $\mathbf{X}$ denotes a third-order variable;

 (5) and lastly, the law of excluded middle $\varphi \vee \neg\varphi$ for all arithmetic formulae φ, i.e. formulae that do not contain second- or third-order quantifiers.

Our paper will be more concerned with Weaver's formal theory itself than with the philosophical justifications behind it. It is already observed by Rathjen in [6] that his theory is interpretable in the weak set theory $\mathrm{CZF} + \mathrm{LPO} + \mathrm{RDC}$, which is known to have the same proof-theoretic ordinal as CZF itself, namely the Bachmann–Howard ordinal. However, we will demonstrate a much lower upper bound for its strength in section 2, where a realisability argument is constructed to show that

Theorem 1.1. *The classical fragment* Σ_1^1-AC *of second-order arithmetic (with full induction) interprets Weaver's system* CM.

This implies that the proof-theoretic ordinal of CM is precisely $\varphi_{\varepsilon_0}(0)$.

In section 3, we will look at the implications of extending Weaver's system by asserting that all second-order objects form a global well-ordering, which Weaver proposed originally in section 2.3 of [10] but did not discuss in detail afterwards. The rest of the paper will then be concerned with determining the effect of possible extensions on the proof-theoretic strength of the theory: in section 4, we will show that the classical theory BI of bar induction is required to interpret a full axiomatisation of the global well-ordering, rendering its proof-theoretic ordinal as the Bachmann–Howard ordinal $\theta_{\varepsilon_{\Omega+1}}(0)$; while in section 5, we will additionally show that if the axiom asserting the countability of the initial segments of this well-ordering is omitted, the same strength cannot be achieved.

2   An ordinal analysis of CM

2.1   The axioms of CM

To begin, we will first make a few clarifications/modifications to Weaver's formal theory, for the convenience of our later analysis. Following Weaver, we still treat the theory CM in a third-order formal language where first-order variables are denoted by lowercase letters $m, n, \ldots$, second-order variables are denoted by italic uppercase letters $X, Y, \ldots$, and third-order variables are denoted by bold uppercase letters $\mathbf{X}, \mathbf{Y}, \ldots$. The membership relation between first-and-second-order and second-and-third-order objects will be distinguished[1] and rendered as $n \in_1 X$ and $X \in_2 \mathbf{X}$.

Non-logical symbols of this language shall consist of all primitive symbols from the language of PA (i.e. zero, successors, addition and multiplication) as well as a binary relation symbol $\prec$ between two second-order variables. This relation will serve as the global well-ordering on second-order objects in the extensions, so for convenience we will enforce that it is present in the formal language from the beginning, though the only axiom in CM concerning it will be to assert that this is a decidable relation:

(6) $\forall X \forall Y (X \prec Y \vee \neg X \prec Y)$.

In the language, we will treat first-order equality as a primitive symbol with the standard equality axioms, while second- and third-order equalities are treated as abbreviations:

$$
\begin{aligned}
X_1 = X_2 &\quad :\Leftrightarrow \quad \forall n\, (n \in_1 X_1 \leftrightarrow n \in_1 X_2), \\
\mathbf{X}_1 = \mathbf{X}_2 &\quad :\Leftrightarrow \quad \forall X\, (X \in_2 \mathbf{X}_1 \leftrightarrow X \in_2 \mathbf{X}_2).
\end{aligned}
$$

Although not mentioned in Weaver's formalisation, we will include the following axioms asserting the extensionality of relations and predicates for second-order variables:

(7) $X = Y \to (X \in_2 \mathbf{X} \leftrightarrow X \in_2 \mathbf{Y})$,

(8) $X = Y \wedge X' = Y' \to (X \prec Y \leftrightarrow X' \prec Y')$.

These are needed in order to show that

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[1] This is done so that, when we work with a realisability model and have natural number realisers for all three orders of variables, we can still read off the order of the realiser from the presentation of the formula directly.

Proposition 2.1. *Given axioms* (7) *and* (8), *the Leibniz law holds, i.e.*

$$t = s \to (\varphi(t) \leftrightarrow \varphi(s)),$$

for any formula φ and any terms t, s of the same, correct order.

Proof. This follows from simple induction on the complexity of φ. Just notice that all atomic cases hold trivially by definition of the equality symbol (which is primitive for first-order terms, but non-primitive for higher-order terms), except when $\varphi(t)$ is $t \in_2 \mathbf{X}$, $t \prec X$ or $X \prec t$, where the extensionality assertions above is used. $\square$

The Leibniz law is necessary for expressions with set-comprehension terms, such as $\{n : \varphi(n)\} \in_2 \mathbf{X}$, to act as non-ambiguous abbreviations in the language.

Additionally, the arithmetic decidability scheme (5) can be complicated to work with at times, so we will break it down into two parts:

(5′) atomic decidability axioms

$$n = m \vee \neg n = m, \qquad n \in_1 X \vee \neg n \in_1 X, \qquad X \in_2 \mathbf{X} \vee \neg X \in_2 \mathbf{X};$$

(5″) the numerical omniscience scheme, also known as the limited principle of omniscience,

$$\forall n \, (\varphi(n) \vee \psi(n)) \to (\forall n \, \varphi(n) \vee \exists n \, \psi(n))$$

for any third-order formulae $\varphi(n)$ and $\psi(n)$.

We know that

Proposition 2.2. *Over the rest of the axioms of* CM, *the axiom scheme* (5) *is equivalent to* (5′) + (5″).

Proof. The forward direction is proved by Weaver as theorem 2.2 in [10]; the backward direction is trivial through an induction on the complexity of an arithmetic formula. $\square$

Therefore, in this paper, we will take CM to formally denote the collection of axioms (1) + (2) + (3) + (4) + (5′) + (5″) + (6) + (7) + (8), each precisely as defined above.

Lastly, the binary pairing function $\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ will always denote a fixed bijection in this paper, e.g. given by

$$\langle m, n \rangle = (m + n) \, (m + n + 1) \, / 2 + m.$$

with inverses $(-)_0, (-)_1 : \mathbb{N} \to \mathbb{N}$. The fact that every natural number is a code for some pair will be convenient for some constructions below. In [10], Weaver notably used a diffferent, non-surjective map $\langle m, n \rangle = 2^m \cdot 3^n$. This difference does affect the precise meaning of certain formulae, for example the axiom scheme (3) of dependent choice. However, since CM contains classical first-order PA as a subtheory, it can be shown that different interpretations of the pairing function are ultimately equivalent.

2.2 Σ_1^1 function codes

We will set up the realisability structure for CM in a classical context of second-order arithmetic, using ACA_0 as the base theory. Following definition VIII.3.18 in Simpson [7], we let $\pi(e, m, a, b, X)$ denote a universal Π_1^0-formula with all free variables indicated, and correspondingly a universal Σ_1^1-formula is given by $\exists X\,\pi(e, m, a, b, X)$, i.e. for any Σ_1^1-formula $\varphi(m, a, b)$ with all free variables indicated, ACA_0 proves that there exists $e \in \mathbb{N}$ such that

$$\forall m\,\forall a\,\forall b\,(\varphi(m, a, b) \leftrightarrow \exists X\,\pi(e, m, a, b, X)).$$

We shall use the notation

$$\{e\}(a) = b \quad :\Leftrightarrow \quad \exists X\,\pi(e_0, e_1, a, b, X).$$

We say that $\{e\}(a)$ *converges*, denoted $\{e\}(a)\!\downarrow$, when $\exists! b\,\{e\}(a) = b$ holds. We write

$$\begin{aligned}
\mathrm{dom}(e) &:= \{a : \{e\}(a)\!\downarrow\}, \\
\mathrm{ran}(e, X) &:= \{b : \exists a \in X\,\{e\}(a) = b\}, \\
[X, Y] &:= \{e : X \subseteq \mathrm{dom}(e) \wedge \mathrm{ran}(e, X) \subseteq Y\}
\end{aligned}$$

for any definable classes $X, Y \subseteq \mathbb{N}$. Note that $\mathrm{dom}(e)$ is Σ_2^1 and hence so is $[X, Y]$, even when X and Y are both arithmetic subsets of $\mathbb{N}$.

It follows immediately that

$$\varphi_0(m, a, b) \quad :\Leftrightarrow \quad \{a_0\}(a_1) = b$$

is a (parameterless) Σ_1^1-formula and correspondingly, we can find $e \in \mathbb{N}$ so that for any $f \in \mathbb{N}$ and $a \in \mathrm{dom}(f)$, we have $\langle f, a \rangle \in \mathrm{dom}(\langle e, 0 \rangle)$ and $\{\langle e, 0 \rangle\}(\langle f, a \rangle) = \{f\}(a)$.

In fact, we can prove the following parametrised generalisation:

Proposition 2.3. *Given any definable class A of natural numbers and a Σ_1^1-formula $\varphi(m, a, b)$ such that, for any $a \in A$, there exists a unique $b \in \mathbb{N}$ satisfying $\varphi(a_0, a_1, b)$, then there exists e such that for any $a \in A$, we have $a_0 \in \mathrm{dom}(e)$, $a_1 \in \mathrm{dom}(\{e\}(a_0))$ and*

$$\varphi(a_0, a_1, \{\{e\}(a_0)\}(a_1)).$$

Proof. Simply take $f, n \in \mathbb{N}$ such that

$$\begin{aligned}
\forall m\,\forall a\,\forall b\,(b = \langle m, a \rangle \leftrightarrow \exists X\,\pi(f, m, a, b, X)), \\
\forall m\,\forall a\,\forall b\,(\varphi(m, a, b) \leftrightarrow \exists X\,\pi(n, m, a, b, X)).
\end{aligned}$$

If follows that $\{\{\langle f, n \rangle\}(m)\}(a) = b$ iff $\{\langle n, m \rangle\}(a) = b$ iff $\varphi(m, a, b)$. $\qquad\square$

Corollary 2.4. *The set of natural numbers $\mathbb{N}$ form a partial combinatory algebra (with arithmetic)[2] under the application operation $\{-\}(-)$.*

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[2] By a *partial combinatory algebra with arithmetic*, we require that the distinguished constants $k, s, 0, \mathrm{succ}, \mathrm{pred}, d, p, p_0, p_1$ are defined in the structure for, respectively, the two basic combinators, zero, successor and predecessor on $\mathbb{N}$, definition by cases and the pairing and unpairing functions. Such PCAs are first considered by Feferman in [3] as part of a stronger system for explicit mathematics.

Proof. The only non-trivial tasks are to define the basic combinators k and s. For k, we directly invoke Proposition 2.3 to compute some $k \in \mathbb{N}$ such that for every $a \in \mathbb{N}$ and $b \in \mathrm{dom}(a)$, we have $a \in \mathrm{dom}(k)$, $b \in \mathrm{dom}(\{k\}(a))$ and

$$\{\{k\}(a)\}(a) = \{a\}(b)$$

holds. For s, we first use Proposition 2.3 to compute $s' \in \mathbb{N}$ such that for every $a, b \in \mathbb{N}$ and $c \in \mathrm{dom}(a) \cap \mathrm{dom}(b)$ such that $\{b\}(c) \in \mathrm{dom}(\{a\}(c))$, we have $\langle a, b \rangle \in \mathrm{dom}(s')$, $c \in \mathrm{dom}(\{s'\}(\langle a, b \rangle))$ and

$$\big\{\{s'\}(\langle a, b \rangle)\big\}(c) = \{\{a\}(c)\}(\{b\}(c))$$

holds. Lastly, we use Proposition 2.3 again to compute $s \in \mathbb{N}$ such that for any $a, b \in \mathbb{N}$ (where at least one aforementioned c exists), we have $a \in \mathrm{dom}(s)$, $b \in \mathrm{dom}(\{s\}(a))$ and

$$\{\{s\}(a)\}(b) = \big\{s'\big\}(\langle a, b \rangle). \qquad \square$$

2.3 Realisability of CM

We shall show that the axiom system CM, as defined in subsection 2.1, can be interpreted in Σ_1^1-AC (with unrestricted induction). This interpretation will be set-up as a realisability model where assignments for first-order variables range over $\mathbb{N}$, assignments for second-order variables range over $S_2 = [\mathbb{N}, \{0, 1\}]$, and assignments for third-order variables range over

$$S_3 = \{d \in [S_2, \{0, 1\}] : \forall e, f \in S_2 \, (e \equiv f \to \{d\}(e) = \{d\}(f))\}\,,$$

where the binary relation $\equiv$ on S_2 is defined as

$$e \equiv f \quad :\Leftrightarrow \quad \forall n \, \{e\}(n) = \{f\}(n).$$

Fix an arbitrary interpretation $r \in [S_2 \times S_2, \{0, 1\}]$ for the relation $\prec$, also satisfying

$$\forall e, e', f, f' \in S_2 \, \big(e \equiv f \wedge e' \equiv f' \to \{r\}(\langle e, e' \rangle) = \{r\}(\langle f, f' \rangle)\big)\,,$$

for example just $r = \lambda n.0$, we shall recursively define the following realisability statements:

$$
\begin{aligned}
d \Vdash t = s \quad &:\Leftrightarrow \quad t = s \text{ for any arithmetic terms } t \text{ and } s, \\
d \Vdash t \in_1 e \quad &:\Leftrightarrow \quad \{e\}(t) = 1 \text{ for any arithmetic terms } t, \\
d \Vdash e \in_2 f \quad &:\Leftrightarrow \quad \{f\}(e) = 1, \\
d \Vdash e \prec f \quad &:\Leftrightarrow \quad \{r\}(\langle e, f \rangle) = 1, \\
d \Vdash \varphi \wedge \psi \quad &:\Leftrightarrow \quad d_0 \Vdash \varphi \wedge d_1 \Vdash \psi, \\
d \Vdash \varphi \vee \psi \quad &:\Leftrightarrow \quad (d_0 = 0 \wedge d_1 \Vdash \varphi) \vee (d_0 = 1 \wedge d_1 \Vdash \psi)\,, \\
d \Vdash \neg\varphi \quad &:\Leftrightarrow \quad \forall e \, \neg e \Vdash \varphi, \\
d \Vdash \varphi \to \psi \quad &:\Leftrightarrow \quad \forall e \, (e \Vdash \varphi \to e \in \mathrm{dom}(d) \wedge \{d\}(e) \Vdash \psi)\,, \\
d \Vdash \forall x \, \varphi(x) \quad &:\Leftrightarrow \quad d \in [\mathbb{N}, \mathbb{N}] \wedge \forall n \, \{d\}(n) \Vdash \varphi(n), \\
d \Vdash \exists x \, \varphi(x) \quad &:\Leftrightarrow \quad d_1 \Vdash \varphi(d_0), \\
d \Vdash \forall X \, \varphi(X) \quad &:\Leftrightarrow \quad d \in [S_2, \mathbb{N}] \wedge \forall n \in S_2 \, \{d\}(n) \Vdash \varphi(n), \\
d \Vdash \exists X \, \varphi(X) \quad &:\Leftrightarrow \quad d_0 \in S_2 \wedge d_1 \Vdash \varphi(d_0), \\
d \Vdash \forall \mathbf{X} \, \varphi(\mathbf{X}) \quad &:\Leftrightarrow \quad d \in [S_3, \mathbb{N}] \wedge \forall n \in S_3 \, \{d\}(n) \Vdash \varphi(n), \\
d \Vdash \exists \mathbf{X} \, \varphi(\mathbf{X}) \quad &:\Leftrightarrow \quad d_0 \in S_3 \wedge d_1 \Vdash \varphi(d_0).
\end{aligned}
$$

For any third-order formula $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m, \mathbf{X}_1, \ldots, \mathbf{X}_r)$ with all free variables indicated, we write $d \Vdash \varphi$ to denote

$$\forall a_1, \ldots, a_n \in \mathbb{N} \, \forall b_1, \ldots, b_m \in S_2 \, \forall c_1, \ldots, c_r \in S_3$$
$$\{d\}(\langle a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_r \rangle) \Vdash \varphi(a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_r).$$

We shall now show in Σ_1^1-AC that whenever

$$\mathrm{CM} \vdash \varphi(x_1, \ldots, x_n, X_1, \ldots, X_m, \mathbf{X}_1, \ldots, \mathbf{X}_r),$$

φ is realised in the sense above.

Firstly, it is easy to verify that our realisability conditions interpret intuitionistic logic. Take note of the following before we begin:

- Since we work in Σ_1^1-AC, the pointclass Σ_1^1 is closed under adding any first-order quantifications.

- Here we treat third-order logic simply as a 3-sorted language in first-order logic, therefore all logical axioms and rules of inferences have to be verified for all three sorts. Luckily, due to the way we defined the realisability conditions above, verification for all the sorts are entirely analogous.

Now, we use the same Hilbert-style formal system as Weaver mentioned in [10]. By Corollary 2.4, the realisers can simply be given as λ-constructs:

- The axiom $\varphi \to (\psi \to \varphi)$ is realised[3] by $\lambda p.\lambda e.\lambda f.e$;

- $(\varphi \to \psi) \to ((\varphi \to (\psi \to \sigma)) \to (\varphi \to \sigma))$ is realised by

$$\lambda p.\lambda f.\lambda g.\lambda e.\{\{g\}(e)\}(\{f\}(e));$$

- $\varphi \to (\psi \to \varphi \wedge \psi)$ is realised by $\lambda p.\lambda e.\lambda f. \langle e, f \rangle$;

- $\varphi \wedge \psi \to \varphi$ is realised by $\lambda p.\lambda e.e_0$;

- $\varphi \wedge \psi \to \psi$ is realised by $\lambda p.\lambda e.e_1$.

- $\varphi \to \varphi \vee \psi$ is realised by $\lambda p.\lambda e. \langle 0, e \rangle$;

- $\psi \to \varphi \vee \psi$ is realised by $\lambda p.\lambda e. \langle 1, e \rangle$;

- $(\varphi \to \sigma) \to ((\psi \to \sigma) \to (\varphi \vee \psi \to \sigma))$ is realised by

$$\lambda p.\lambda f.\lambda g.\lambda n. \begin{cases} \{f\}(n_1) & \text{if } n_0 = 0, \\ \{g\}(n_1) & \text{otherwise;} \end{cases}$$

- $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$ is realised simply by $\lambda p.\lambda f.\lambda g.0$. This is a correct realiser because, suppose that $f \Vdash \varphi \to \psi$, $g \Vdash \varphi \to \neg\psi$, $d \Vdash \varphi$, then $\{g\}(d) \Vdash \neg\psi$ while $\{f\}(d) \Vdash \psi$, a contradiction. Therefore, no such realiser d exists and 0 — in fact, anything — realises $\neg\varphi$.

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[3] Here and in all subsequent realisability claims, the first parameter p accommodates for the tuple of assignments $\langle a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_r \rangle$ to free variables in the axiom.

- $\varphi(t) \to \exists x\, \varphi(x)$ is realised[4] by $\lambda p.\lambda e.\langle t, e\rangle$, where parameters in p may be used when t is a complex first-order term;

- $\forall x\, \varphi(x) \to \varphi(t)$ is realised by $\lambda p.\lambda f.\{f\}(t)$, where again parameters in p may be used when computing t.

- Regarding the equality axioms for the primitive first-order equality symbol, it suffices to realise $t = t$ by $\lambda p.0$ and $t = s \wedge \varphi(t) \to \varphi(s)$ by $\lambda p.\lambda e.e_1$. The latter works since it is easy to show by induction that, when t and s are actually equal terms in PA, $d \Vdash \varphi(t)$ implies $d \Vdash \varphi(s)$ as well.

For the rules of inferences:

MP Given $e \Vdash \varphi$ and $f \Vdash \varphi \to \psi$, the conclusion ψ is realised by $\lambda p.\{\{f\}(p)\}(\{e\}(p))$.

$\forall$-**Gen** Given $f \Vdash \varphi \to \psi$, where x does not occur in φ, the conclusion $\varphi \to \forall x\, \psi$ is realised by

$$\lambda p.\lambda d.\lambda n.\{\{f\}(p, n)\}(d),$$

where by $\{f\}(p, n)$ we mean that n is passed as an assignment to the free variable x in $\varphi \to \psi$.

$\exists$-**Gen** Given $f \Vdash \varphi \to \psi$, where x does not occur in ψ, the conclusion $\exists x\, \varphi \to \psi$ is similarly realised by

$$\lambda p.\lambda d.\{\{f\}(p, d_0)\}(d_1).$$

It suffices now to further show that the non-logical axioms of CM are realised. To begin, the number axioms (1) in CM are just true, quantifier-free formulae in PA (with free variables), which can be easily realised by any Σ_1^1 function with appropriate domain and range.

For the schemes (2) and (3), we will need the recursion lemma for Σ_1^1 functions:

Lemma 2.5 (Σ_1^1-AC)**.** *There exists $f \in \mathbb{N}$ such that, consider any definable class A of natural numbers and $e \in [A, \mathbb{N}]$ such that for any $a \in A$, $\{e\}(a) \in (A)_{a_0+1}$, then for any $z \in (A)_0$, we have $\langle e, z\rangle \in \mathrm{dom}(f)$, $d = \{f\}(\langle e, z\rangle) \in [\mathbb{N}, \mathbb{N}]$ and*

$$\{d\}(0) = z \wedge \forall n\, (\{d\}(n) \in (A)_n \wedge \{d\}(n + 1) = \{e\}(\langle n, \{d\}(n)\rangle)).$$

Proof. The desirable function computed by d is given by

$$\eta(n; e, z) = y \quad :\Leftrightarrow \quad \exists s \in \mathrm{Seq}\, (s_0 = z \wedge s_n = y \wedge \forall i < n\, \{e\}(\langle n, s_i\rangle) = s_{i+1}).$$

where Seq is the class of all finite sequences coded in $\mathbb{N}$. Therefore, we can find some $e_0 \in \mathbb{N}$ so that $\eta(-; e, z)$ is computed by Σ_1^1 function code $\langle e_0, \langle e, z\rangle\rangle$, and by Proposition 2.3, we can find f to compute $\lambda a.\langle e_0, a\rangle$.

Now, it follows from induction that f does have the desired property. Namely, if A is Π_{k+1}^1, then Σ_{k+2}^1-induction shows that $\forall n\, \{\{f\}(\langle e, z\rangle)\}(n) \in (A)_n$. $\qquad\square$

Proposition 2.6. *The induction scheme (2) is realised:*

$$\varphi(0) \wedge \forall n\, (\varphi(n) \to \varphi(n + 1)) \to \forall n\, \varphi(n).$$

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[4]Here t can be a term in any of the three sorts, and (if t is a first-order term) it may contain free parameters. The same applies to cases below.

Proof. For any $d \Vdash \varphi(0)$ and $f \Vdash \forall n\, (\varphi(n) \to \varphi(n+1))$, a function $g \Vdash \forall n\, \varphi(n)$ can be given recursively:

$$\{g\}(0) = d,$$
$$\{g\}(n+1) = \{\{f\}(n)\}(\{g\}(n)).$$

In this recursion, we will need to check $\{g\}(n) \Vdash \varphi(n)$ for any n and this follows from unrestricted[5] induction. $\qquad\square$

Proposition 2.7. *The recursion scheme* (3) *is realised:*

$$\forall n\, \forall X\, \exists Y\, \varphi(n, X, Y) \to \forall X\, \exists Z\, \left((Z)_0 = X \land \forall n\, \varphi\big(n, (Z)_n, (Z)_{n+1}\big) \right).$$

Proof. For any $f \Vdash \forall n\, \forall X\, \exists Y\, \varphi(n, X, Y)$ and any $x \in S_2$, we first compute some function g recursively:

$$\{g\}(0) = x,$$
$$\{g\}(n+1) = \{\{f\}(n)\}(\{g\}(n))_0.$$

The desired Z is then constructed as some $z \in S_2$ given by $\{z\}(\langle m, n \rangle) = \{\{g\}(m)\}(n)$. Notice that $(Z)_0 = X$ is realised by any function with appropriate domain and range because we are working with implications like $\langle 0, x \rangle \in_1 Z \to x \in_1 X$, where the atoms are realised by anything as long as they are true. The conditions $\varphi\big(n, (Z)_n, (Z)_{n+1}\big)$ are realised[6] by $\{\{f\}(n)\}(\{g\}(n))_1$. $\qquad\square$

The decidable comprehension axioms (4) are easy cases here:

Proposition 2.8. *The following comprehension axioms are realised:*

$$\forall n\, (\varphi(n) \lor \neg\varphi(n)) \to \exists X\, \forall n\, (n \in_1 X \leftrightarrow \varphi(n)),$$
$$\forall X\, (\varphi(X) \lor \neg\varphi(X)) \to \exists \mathbf{X}\, \forall X\, (X \in_2 \mathbf{X} \leftrightarrow \varphi(X)).$$

Proof. For any $f \Vdash \forall n\, (\varphi(n) \lor \neg\varphi(n))$, X can simply be constructed as some $x \in S_2$ such that $\{x\}(n) = 1 - \{f\}(n)_0$. Observe that $n \in_1 X \to \varphi(n)$ is realised by the function $\lambda a.\{f\}(n)_1$, where the opposite $\varphi(n) \to n \in_1 X$ is realised by an arbitrary function.

The third-order case is entirely analogous. $\qquad\square$

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[5]Even though we never use an arbitrary amount of second-order quantifiers, the use of unrestrict induction is necessary here: suppose that φ contains an alternating sequence of quantifiers, then the statement $d \Vdash \varphi(p)$ can expands to a form like

$$d \in [\mathbb{N}, \mathbb{N}] \land \forall n_0\, \exists d_1 \in [\mathbb{N}, \mathbb{N}]\, \big(\{d\}(n_0)_1 = d_1 \land \forall n_1\, \exists d_2 \in [\mathbb{N}, \mathbb{N}] \ldots \exists d_i \in [\mathbb{N}, \mathbb{N}] \ldots$$

Here the class $[\mathbb{N}, \mathbb{N}] = \{e : \forall a\, \exists! b\, \{e\}(a) = b\}$ is Σ_2^1 but lies within a sequence of first-order quantifiers. Since Σ_1^1-AC does not show that the pointclass Σ_2^1 is closed under adding first-order quantifiers, it cannot prove that this statement has bounded complexity.

[6]To be completely formal, we may understand $\varphi\big(n, (Z)_n, (Z)_{n+1}\big)$ as

$$\exists X = (Z)_n\, \exists Y = (Z)_{n+1}\, \varphi(n, X, Y),$$

so the formula is realised by $\langle \{g\}(n), \langle *, \langle \{g\}(n+1), \langle *, \{\{f\}(n)\}(\{g\}(n))_1 \rangle \rangle \rangle \rangle$, where $*$ can be any function code with appropriate domain and range.

For the atomic decidability axioms (5′):

- $t = s \vee \neg t = s$ is realised by

$$\lambda p. \begin{cases} \langle 0, 0 \rangle & \text{if } t = s, \\ \langle 1, 0 \rangle & \text{otherwise,} \end{cases}$$

  where p represents free parameters used to compute t and s;

- $t \in_1 e \vee \neg t \in_1 e$ is realised by $\lambda \langle p, e \rangle. \langle 1 - \{e\}(t), 0 \rangle$ where p represents free parameters in t;

- $e \in_2 f \vee \neg e \in_2 f$ is realised by $\lambda \langle e, f \rangle. \langle 1 - \{f\}(e), 0 \rangle$.

And likewise for the decidability of the order relation (6): $e \prec f \vee \neg e \prec f$ is realised by $\lambda \langle e, f \rangle. \langle 1 - \{r\}(\langle e, f \rangle), 0 \rangle$.

Proposition 2.9. *The numerical omniscience scheme (5″) is realised:*

$$\forall n \, (\varphi(n) \vee \psi(n)) \rightarrow (\forall n \, \varphi(n) \vee \exists n \, \psi(n)) .$$

Proof. We first fix some $g \in [\mathbb{N}, \mathbb{N}]$ that computes $\lambda f. \lambda n. \{f\}(n)_1$. Now construct the function

$$\{e\}(f) = d \ :\Leftrightarrow \ (d_0 = 0 \wedge \forall n \, \{f\}(n)_0 = 0 \wedge d_1 = \{g\}(f)) \vee$$
$$(d_0 = 1 \wedge \{f\}(d_{10})_0 = 1 \wedge \forall n < d_{10} \, \{f\}(n)_0 = 0 \wedge d_{11} = \{f\}(d_{10})_1) .$$

Observe that since the base theory satisfies classical logic, a unique result d is always well-defined for any $f \Vdash \forall n \, (\varphi(n) \vee \psi(n))$. Then $\lambda p.e$ is a realiser for the desired axiom. $\qquad \square$

Finally, observe that in our set-up, we have ensured for any $d \in S_3$,

$$\forall e, f \in S_2 \, (e \equiv f \rightarrow \{d\}(e) = \{d\}(f)) ,$$

and likewise for the fixed interpretation r of the relation $\prec$,

$$\forall e, e', f, f' \in S_2 \, (e \equiv f \wedge e' \equiv f' \rightarrow \{r\}(\langle e, e' \rangle) = \{r\}(\langle f, f' \rangle)) .$$

Thus the extensionality assertions (7) and (8) will be realised by any Σ_1^1 function with appropriate domain and range.

We have so far argued that assuming Σ_1^1-AC, all the rules of inferences of intuitionistic logic, all logical axioms and all non-logical axioms of CM are realised. The following theorem thus follows from simple induction on the length of derivations:

Theorem 2.10 (Σ_1^1-AC). *Whenever*

$$\text{CM} \vdash \varphi(x_1, \ldots, x_n, X_1, \ldots, X_m, \mathbf{X}_1, \ldots, \mathbf{X}_r),$$

there exists $d \in \mathbb{N}$ such that $d \Vdash \varphi$.

2.4　An ordinal analysis of CM

Let Σ_1^1-AC and Σ_1^1-DC denote the axioms in ACA plus the axioms of Σ_1^1-choice and Σ_1^1-dependent choice respectively, and let Σ_1^1-AC^i denote the intuitionistic fragment of Σ_1^1-AC. It is well-known that:

Theorem 2.11. *The following theories have the same proof-theoretic strength:*

$$\Sigma_1^1\text{-AC}^i \equiv \Sigma_1^1\text{-AC} \equiv \Sigma_1^1\text{-DC}.$$

Here, the proof-theoretic equivalence of Σ_1^1-AC^i and Σ_1^1-AC is due to Aczel [1]. Friedman showed in [4] that one can interpret both Σ_1^1-AC and Σ_1^1-DC in Δ_1^1-CA, and thus the three theories are equivalent in strength since Δ_1^1-CA is implied by Σ_1^1-AC.

Since Σ_1^1-AC^i is an obvious subtheory of CM and our analysis in previous sections shows that CM can be interpreted in Σ_1^1-AC, it is an immediate corollary that:

Corollary 2.12. CM $\equiv \Sigma_1^1$-DC. *In particular, since the exact proof-theoretic strength of Σ_1^1-DC is analysed in [5], we have* $|\text{CM}| = \left|\Sigma_1^1\text{-DC}\right| = \varphi_{\varepsilon_0}(0)$.

3　Axiomatising a global well-ordering

3.1　Axioms of a global well-ordering

As mentioned in the introduction, in section 2.3 of [10], Weaver suggested briefly that one could extend the system CM by adding a "genuine choice axiom", stating that all the second-order objects in the theory are constructed in well-ordered stages. Formally, this can be represented as the following axioms:

(W1)　the binary relation $\prec$ between second-order objects is a linear ordering;

(W2)　the transfinite induction scheme

$$\forall X\,(\forall Y\,(Y \prec X \to \varphi(Y)) \to \varphi(X)) \to \forall X\,\varphi(X),$$

　　　for a certain class of formulae;

(W3)　the axiom of countable initial segments

$$\forall X\,\exists Z\,\forall Y\,(Y \prec X \to \exists n\,Y = (Z)_n).$$

Do note that for the axiom scheme (W2) of transfinite induction, Weaver had been explicit that he did not accept the scheme for an arbitrary third-order formula φ in the language, since it is obviously impredicative to assert that there exists a desired well-ordering $\prec$, on which transfinite induction holds for some formula φ which involves the ordering relation $\prec$ itself. This is deemed unjustified according to his philosophical standpoint of mathematical conceptualism.

I plan to not delve deep into Weaver's philosophical arguments here, but one must point out that he failed to formally describe a class of formulae on which the axioms can characterise a restricted, predicative extension of CM; and I believe it still remains open whether such a nice fragment of the full, unrestricted axiom scheme of transfinite induction can be found. More discussion on this will be reserved till the end in section 6.

For now, we shall still stick to the entire collection of axioms (W1) + (W2) + (W3) where the formula φ in the transfinite induction scheme ranges over all third-order formulae in the language, henceforth denoted GWO. In section 4, we will give an ordinal analysis of the extension CM + GWO, which will also serve as an upper bound of the proof-theoretic strength of any of its predicative fragments, if they actually exist.

3.2 Zorn's lemma and applications

Before the ordinal analysis, we shall note that while in [10] Weaver gave a very detailed exposition of how ordinal mathematics, including theorems of real analysis, topology and functional analysis, can be treated inside the intuitionistic formal theory CM, in order to justify its viability as a foundational theory of mathematics, he never did the same with this proposed extension, other than a short comment after proving the (separable) Hahn–Banach theorem:

Theorem (CM). *Let $\mathbf{E}$ be a separable Banach space and $\mathbf{E}_0$ be a separable closed subspace. Then any bounded linear functional $\mathbf{f}_0 : \mathbf{E}_0 \to \mathbf{R}$ can be extended to a bounded linear functional $\mathbf{f} : \mathbf{E} \to \mathbf{R}$ satisfying $\|\mathbf{f}\| = \|\mathbf{f}_0\|$.*

Here, Weaver claimed that using the axioms for the global well-ordering $\prec$, this can be proved without assuming that $\mathbf{E}$ is separable through essentially the same proof.

In this section, we will expand on the direction by showing that a general result analogous to Zorn's lemma can be proved in CM + GWO; thus the global well-ordering axioms indeed serve their purpose as a choice principle and constitute indeed a useful extension to consider if one wants to use CM as a foundational theory.

To begin, we will first observe that omniscience holds for a bounded second-order quantifier:

Lemma 3.1 (CM + GWO). *For any formula $\varphi(X)$ and $\psi(X)$, we have*

$$\forall X \left(\forall Y \prec X \left(\varphi(Y) \vee \psi(Y) \right) \to \left(\forall Y \prec X \, \varphi(Y) \vee \exists Y \prec X \, \psi(Y) \right) \right).$$

Obviously the same also holds when every $\prec$ is replaced by $\preceq$ in the proposition.

Proof. The axiom of countable initial segments asserts that for any second-order set X, there exists Z satisfying

$$\forall Y \prec X \, \exists n \, Y = (Z)_n.$$

Assuming that $\forall Y \prec X \left(\varphi(Y) \vee \psi(Y) \right)$, then by numerical omniscience we have

$$\forall n \left((Z)_n \prec X \to \varphi((Z)_n) \right) \vee \exists n \left((Z)_n \prec X \wedge \psi((Z)_n) \right).$$

The former disjunct implies $\forall Y \prec X \, \varphi(Y)$, while the latter disjunct implies $\exists Y \prec X \, \psi(Y)$. $\qquad\square$

We also need the fact that countable collections of second-order objects can be represented as one single second-order object in CM. Instead of the obvious coding that uses slices $(Z)_n$, we will use a slightly modified construction so that we can accommodate the empty collection as well: for any second-order set X, let $[X]_n =$

$\{m : \langle n, 0 \rangle \in_1 X \wedge \langle n, m+1 \rangle \in_1 X\}$, and define a new binary relation $\in^*$ between *two* second-order objects as

$$Y \in^* X \quad :\Rightarrow \quad \exists n \left(\langle n, 0 \rangle \in_1 X \wedge Y = [X]_n\right).$$

We have some supplementary abbreviations such as

$$X \subseteq^* Y \quad :\Rightarrow \quad \forall Z \in^* X \; Z \in^* Y,$$
$$X \subseteq^* \mathbf{X} \quad :\Rightarrow \quad \forall Z \in^* X \; Z \in_2 \mathbf{X},$$
$$X =^* Y \quad :\Rightarrow \quad X \subseteq^* Y \wedge Y \subseteq^* X$$

to compare these countable collections. One immediately observes that the omniscience statement for countable collections,

$$\forall Y \in^* X \left(\varphi(Y) \vee \psi(Y)\right) \to \left(\forall Y \in^* X \; \varphi(Y) \vee \exists Y \in^* X \; \psi(Y)\right),$$

follows directly from numerical omniscience, thus the relations defined above are all decidable.

In the same way, we can also use certain set-theoretic expressions over second-order objects as second-order terms. For example, we may have

$$\{X_1, \ldots, X_k\} := \{\langle n, m \rangle : 1 \le n \le k \wedge (m = 0 \vee \exists i \in_1 X_n \; m = i + 1)\},$$
$$\bigcup X := \{\langle \langle n, m \rangle, i \rangle : \langle n, 0 \rangle \in_1 X \wedge \langle n, \langle m, i \rangle + 1 \rangle \in_1 X\},$$
$$\{Y \in^* X : \varphi(Y)\} := \{\langle n, m \rangle : \langle n, 0 \rangle \in_1 X \wedge \langle n, m \rangle \in_1 X \wedge \varphi([X]_n)\},$$

where in the last definition, the formula $\varphi(Y)$ must be decidable. It is easy to verify that these collections "contain" the same second-order sets as their third-order counterparts.

We can now state our analogue of Zorn's lemma:

Proposition 3.2 (CM + GWO). *Fix a third-order set* $\mathbf{A}$ *and let* $\varphi(X)$ *be a decidable formula (possibly with other parameters) on the countable subsets of* $\mathbf{A}$, *i.e. we have* $\forall X \subseteq^* \mathbf{A} \left(\varphi(X) \vee \neg\varphi(X)\right)$. *If* φ *as a predicate is downward closed, i.e.*

$$\forall X, Y \left((\varphi(X) \wedge Y \subseteq^* X) \to \varphi(Y)\right),$$

and also closed under unions of countable chains, i.e.

$$\forall X \left(\left(\forall Y, Z \in^* X \left(Y \subseteq^* Z \vee Z \subseteq^* Y\right) \wedge \forall Y \in^* X \; \varphi(Y)\right) \to \varphi\left(\bigcup X\right)\right),$$

then there exists some $\mathbf{X} \subseteq \mathbf{A}$ *satisfying* $\forall X \subseteq^* \mathbf{X} \; \varphi(X)$, *and is maximal among such subsets in the sense that*

$$\forall Y \left(Y \notin \mathbf{X} \to \exists Z \subseteq^* \mathbf{X} \; \neg\varphi(Z \cup \{Y\})\right).$$

Proof. Let

$$\psi(X, Z) \quad := \quad \varphi(Z) \wedge (\varphi(Z \cup \{X\}) \to X \in^* Z),$$
$$\theta(X, Z) \quad := \quad \forall Y \in^* Z \; Y \preceq X \wedge \forall Y \preceq X \; \psi(Y, \{W \in^* Z : W \preceq Y\}).$$

We first show by transfinite induction that

$$\forall X, Z_1, Z_2 \left((\theta(X, Z_1) \wedge \theta(X, Z_2)) \to Z_1 =^* Z_2 \right).$$

This is because for any $Y \prec X$, $\theta(X, Z_1)$ implies $\theta(Y, \{W \in^* Z_1 : W \preceq Y\})$ and likewise for Z_2. Therefore, from the inductive hypothesis we know that

$$\{W \in^* Z_1 : W \preceq Y\} =^* \{W \in^* Z_2 : W \preceq Y\},$$

which implies $Y \in^* Z_1 \leftrightarrow Y \in^* Z_2$. Now assume that $X \in^* Z_1$, then $Z_1 =^* Z_2 \cup \{X\}$, so $\varphi(Z_2 \cup \{X\})$ holds and thus $X \in^* Z_2$. The same can be said assuming $X \in^* Z_2$, so we can finally conclude that $Z_1 =^* Z_2$.

We then show by transfinite induction again that $\forall X \, \exists Z \, \theta(X, Z)$. For any X, let W be such that $\forall Y \prec X \, \exists n \, Y = (W)_n$, and the inductive hypothesis implies

$$\forall n \, \exists Z \left((W)_n \prec X \to \theta((W)_n, Z) \right).$$

By dependent choice, we find some U satisfying $\forall n \, ((W)_n \prec X \to \theta((W)_n, (U)_n))$. Now define

$$Z' = \{\langle n, m \rangle : (W)_n \prec X \wedge (m = 0 \vee \exists i \in_1 (U)_n \, m = i + 1)\},$$

then for each $Y \in^* Z'$, we have $\varphi(Y)$. Here, Z' is additionally a chain because for any $[Z']_m, [Z']_n \in^* Z'$, we can assume without loss of generality that $(W)_m \preceq (W)_n$, and we will have

$$[Z']_m =^* \{W \in^* [Z']_n : W \preceq (W)_m\} \subseteq^* [Z']_n$$

from the previous induction argument. Hence $\varphi(\bigcup Z')$. It also follows that $[Z']_n = \{Y \in^* \bigcup Z' : Y \prec (W)_n\}$ for any n such that $(W)_n \prec X$.

Therefore, it suffices to set $Z = (\bigcup Z') \cup \{X\}$ if $\varphi((\bigcup Z') \cup \{X\})$ holds and $Z = \bigcup Z'$ otherwise (which is possible because φ is decidable). It is easy to verify that $\theta(X, Z)$ holds indeed.

Finally, we observe that $\theta(X, Z)$ is decidable by Lemma 3.1. Therefore, we can comprehend the diagonal set

$$\mathbf{X} = \{Y : \exists Z \left(\theta(Y, Z) \wedge Y \in^* Z \right)\}.$$

For any countable subset $X \subseteq^* \mathbf{X}$, we have

$$\forall n \, \exists W \left(\langle n, 0 \rangle \in_1 X \to \forall Y \preceq [X]_n \, \exists m \, Y = (W)_m \right),$$

and by dependent choice, we can find some U satisfying

$$\forall n \left(\langle n, 0 \rangle \in_1 X \to \forall Y \preceq [X]_n \, \exists m \, Y = ((U)_n)_m \right).$$

We can now diagonalise and construct

$$Y_0 = \left\{ \langle n, m \rangle : \langle n, m \rangle \notin_1 ((U)_n)_m \right\}$$

so that $\forall Y \in^* X \, \forall Y' \preceq Y \, Y' \prec Y_0$. Let Z_0 be a countable collection satisfying $\theta(Y_0, Z_0)$, then our definition of $\mathbf{X}$ and the uniqueness of Z_0 as shown previously ensure that for any $Z \prec Y_0$,

$$Y \in_2 \mathbf{X} \leftrightarrow Y \in^* Z_0.$$

In other words, $X \subseteq^* Z_0$, and $\varphi(X)$ follows immediately from $\varphi(Z_0)$.

On the other hand, suppose that some $Y \not\in_2 \mathbf{X}$. Likewise, there exists Z satisfying $\theta(Y, Z)$ and $Y \not\in^* Z$, which, by our stipulation of ψ at the beginning of this proof, imply $\neg\varphi(Z \cup \{Y\})$. The uniqueness of Z ensures that also $Z \subseteq^* \mathbf{X}$. Therefore, we have checked that $\mathbf{X}$ has all the desired properties. $\qquad\square$

This can be applied just like the classical Zorn's lemma, to show that one can find maximal subsets satisfying certain properties in uncountable collections, as long as the corresponding decidability criterion can be verified through intuitionistic reasoning. As an example, let us consider third-order objects $\mathbf{F}$, $\mathbf{V}$ such that $\mathbf{F}$ is a field and $\mathbf{V}$ is an $\mathbf{F}$-vector space, with their respective unary and binary operations also given as third-order objects, following the conventions in section 3 of [10]. We can derive that

Corollary 3.3 (CM + GWO). *Suppose that the formula*

$$\varphi(k, X) \quad :\equiv \quad \exists Y \left(\forall n \le k\, (Y)_n \in \mathbf{F} \wedge \exists n \le k\, (Y)_n \ne 0_{\mathbf{F}} \wedge 0_{\mathbf{V}} = \sum_{n=0}^{k} (Y)_n \cdot (X)_n \right),$$

i.e. informally the statement "$(X)_0, \ldots, (X)_k$ are linearly dependent as vectors in $\mathbf{V}$", is decidable (whenever $(X)_0, \ldots, (X)_k \in_2 \mathbf{V}$), then $\mathbf{V}$ has a basis $\mathbf{X} \subseteq \mathbf{V}$ such that every vector $V \in \mathbf{V}$ is spanned by finitely many elements in $\mathbf{X}$.

Proof. Fix a way to code finite tuples of natural numbers, so that every $a \in \mathbb{N}$ codes some tuple $\left\langle (a)_0, \ldots, (a)_{|a|} \right\rangle$. Now, let

$$\psi(X) \quad :\equiv \quad X \subseteq^* \mathbf{V} \wedge \forall a\, (\exists n \le |a|\, \langle (a)_n, 0 \rangle \not\in_1 X \vee \neg\varphi(|a|, T(X, a)))$$

where the tuple $T(X, a) = \{\langle n, m \rangle : n \le |a| \wedge \langle (a)_n, m + 1 \rangle \in X\}$ collects precisely the elements $[X]_{(a)_0}, \ldots, [X]_{(a)_{|a|}}$. By numerical omniscience, ψ is also decidable.

It is easy to check that the premises of Proposition 3.2 are all satisfied, thus there is a maximal set $\mathbf{X}$ satisfying $\forall X \subseteq^* \mathbf{X}\, \psi(X)$, which also implies $\mathbf{X} \subseteq \mathbf{V}$. Additionally, for any vector $V \in_2 \mathbf{V}$, if $V \in_2 \mathbf{X}$, then V is trivially spanned; otherwise $V \not\in_2 \mathbf{X}$, and we know from decidability that $\neg\psi(Z \cup \{V\})$ for some $Z \subseteq^* \mathbf{X}$, i.e. a finite tuple of vectors from $Z \cup \{V\}$ are linearly dependent. Since Z alone does not span $0_{\mathbf{V}}$, V must be spanned by elements in Z. $\qquad\square$

In general, the criterion $\varphi(k, X)$ may not be decidable for every vector space over every field, due to the second-order quantifier in the beginning. However, very often the field or the vector space in concern is nice enough for the basis to exist:

- When *the field $\mathbf{F}$ is countable*, $\varphi(k, X)$ is decidable simply by numerical omniscience. This shows that, for example, the Hamel basis for the reals $\mathbf{R}$ as a vector space over the rationals $\mathbf{Q}$ exists.

- When *the vector space $\mathbf{V}$ has an inner product* $\langle \cdot, \cdot \rangle : \mathbf{V} \times \mathbf{V} \to \mathbf{F}$, given some finite collection X of vectors in $\mathbf{V}$, we can apply the Gram–Schmidt process by

computing

$$(Y)_n = (X)_n - \sum_{m=0}^{n-1} \langle (Z)_m, (X)_n \rangle (Z)_m,$$

$$(Z)_n = \langle (Y)_n, (Y)_n \rangle^{-1} (Y)_n,$$

for $0 \le n \le k$ and halt when we encounter some $(Y)_n = 0_{\mathbf{V}}$. Obviously $\exists n \le k \, (Y)_n = 0_{\mathbf{V}}$ implies $\varphi(k, X)$; and conversely if we have $0_{\mathbf{V}} = \sum_{n=0}^{k} (W)_n \cdot (X)_n$ where $(W)_n \in_2 \mathbf{F}$ are not all zero, we similarly also have the equality

$$(X)_n = \sum_{m=0}^{n-1} -(W)_m (W)_n^{-1} (X)_m$$

for some $0 \le n \le k$ and thus $(Y)_n = 0_{\mathbf{V}}$ assuming $(Y)_m \neq 0_{\mathbf{V}}$ for all $m < n$. In this case, $\varphi(k, X)$ is equivalent to $\exists n \le k \, (Y)_n = 0_{\mathbf{V}}$ and thus decidable.

We shall sketch that $\varphi(k, X)$ is decidable also when $\mathbf{F}$ *is a complete* $\mathbf{R}$-*normed field with a countable dense set* $C \subseteq^* \mathbf{F}$ *with respect to the induced norm*, and $\mathbf{V}$ *has a compatible norm* $\|\cdot\| : \mathbf{V} \to \mathbf{R}$. Any real or complex Banach space, for example, falls in this category.

Corollary 3.4. *Let* $\mathbf{F}$ *be a complete* $\mathbf{R}$-*normed field with a countable dense set* $C \subseteq^* \mathbf{F}$ *with respect to the induced norm, and* $\mathbf{V}$ *be a* $\mathbf{F}$-*vector space with a compatible norm* $\|\cdot\| : \mathbf{V} \to \mathbf{R}$. *Then* $\mathbf{V}$ *has a basis.*

Proof. We consider the condition that

$$\psi(k, X) \quad :\equiv \quad \exists n \le k \, \forall z \, \exists m_0^{(z)}, \dots, m_{n-1}^{(z)} \left\| (X)_n - \sum_{i=0}^{n-1} [C]_{m_i^{(z)}} (X)_i \right\| < 1/z,$$

which is decidable by numerical omniscience.

First observe that if we have some equality $0_{\mathbf{V}} = \sum_{n=0}^{k} (Y)_n \cdot (X)_n$, by numerical omniscience we can find the last $n \le k$ such that $(Y)_n \neq 0_{\mathbf{F}}$ and rewrite it as

$$(X)_n = \sum_{m=0}^{n-1} -(Y)_m (Y)_n^{-1} (X)_m.$$

The density of C ensures that we can find elements in C close enough to each scalar $-(Y)_m (Y)_n^{-1} \in_2 \mathbf{F}$ so that $\psi(k, X)$ holds.

Now assume instead that $\psi(k, X)$ holds, then by its decidability and induction on the natural numbers we can find the least such $n \le k$. By dependent choice, fix any $0 \le i < n$ and we can collect the sequence of $[C]_{m_i^{(z)}}$ used in ψ for every z. It suffices to show that each such sequence is Cauchy, because then the limits would describe $(X)_n$ as a linear combination of $(X)_0, \dots, (X)_{n-1}$.

Here, the minimality of n will ensure that $(X)_0, \dots, (X)_{n-1}$ are linearly independent, i.e. the function

$$\mathbf{f} : \langle Y_0, \dots, Y_{n-1} \rangle \in \mathbf{F}^n \mapsto \sum_{m=0}^{n-1} Y_m \cdot (X)_m$$

will be injective. With respect to the 1-norm

$$\|\langle Y_0, \dots, Y_{n-1}\rangle\| = \sum_{m=0}^{n-1} \|Y_m\|$$

on $\mathbf{F}^n$, $\mathbf{f}$ is continuous. Thus, if we consider the separable compact subspace $\mathbf{S}^1 = \{Y \in_2 \mathbf{F}^n : \|Y\| = 1\}$, then by proposition 3.40 in [10], the image $\mathbf{f}(\mathbf{S}^1)$ exists and is compact in $\mathbf{V}$. Since $(X)_0, \dots, (X)_{n-1}$ are linearly independent, $0_{\mathbf{V}} \notin_2 \mathbf{f}(\mathbf{S}^1)$, thus there is a minimum $M > 0$ such that $\forall V \in_2 \mathbf{f}(\mathbf{S}^1) \|V\| \geq M$. Now, for any $Y \in_2 \mathbf{F}^n$, $Y \neq 0_{\mathbf{F}^n}$ such that $\|\mathbf{f}(Y)\| < \varepsilon$, we have

$$M \leq \left\| \mathbf{f}\left(\|Y\|^{-1} Y \right) \right\| < \|Y\|^{-1} \varepsilon,$$

i.e. each $\|Y_i\| < M^{-1}\varepsilon$. This justifies precisely what we need, that when the sum $\sum_{i=0}^{n-1} [C]_{m_i^{(z)}} (X)_i$ is Cauchy in the parameter z, then so is the sequence $[C]_{m_i^{(z)}}$ for every fixed i. $\qquad\square$

4 An ordinal analysis of $\mathrm{CM} + \mathrm{GWO}$

4.1 Hyperarithmetic theory in ATR

Now, we will return from our detour on Zorn's lemma back to providing an ordinal analysis of the extended theory $\mathrm{CM}+\mathrm{GWO}$. We shall do so by reusing the realisability model defined in subsection 2.3, but move to a stronger base theory that provides a nice well-ordering on the collection of second-order realisers S_2. This well-ordering will be generated from the hierarchy of hyperarithmetic sets given by iterating Turing jumps.

For now, let us work in the theory ATR (with unrestricted induction). Everything we have done so far using Σ_1^1 functions apply immediately due to the following result, which is theorem V.8.3 in Simpson [7]:

Fact. ATR *implies* Σ_1^1-AC.

The constructions we will introduce here roughly follow section VIII.3 in [7]: we again start from a universal lightface Π_1^0-formula $\pi(e, m, X)$ with all free variables indicated, that is, for any Π_1^0-formula $\varphi(e', m, X)$ with the same number of free variables, ACA_0 proves that

$$\forall e' \exists e \forall m \forall X \left(\varphi(e', m, X) \leftrightarrow \pi(e, m, X) \right).$$

We shall make use of the following series of definitions:

Definition (Relative recursiveness). We say that a set $Y \subseteq \mathbb{N}$ is *X-recursive* and write $Y \leq_T X$ if there exists $e = \langle e_0, e_1 \rangle \in \mathbb{N}$ such that

$$Y = \{m : \pi(e_0, m, X)\} = \{m : \neg\pi(e_1, m, X)\}.$$

We say that e is the *X-recursive index* of Y. We write

$$\mathrm{Rec}^X = \{e \in \mathbb{N} : \forall m \left(\pi(e_0, m, X) \leftrightarrow \neg\pi(e_1, m, X) \right)\}$$

to denote the set of all X-recursive indices, and for any $e \in \mathrm{Rec}^X$, let R_e^X denote the set defined by e. The superscript X here will be omitted when $X = \varnothing$.

Definition (Ordinal notations)**.** Let $\mathcal{O}_+$ denote the set of indices $\alpha \in \mathbb{N}$ where $\alpha_0 \in$ Rec, R_{α_0} is a linear ordering on a set field$(R_{\alpha_0}) \subseteq \mathbb{N}$ and $\alpha_1 \in$ field(R_{α_0}). The canonical partial ordering on $\mathcal{O}_+$ is given by

$$\alpha <_\mathcal{O} \beta \quad :\Leftrightarrow \quad \alpha_0 = \beta_0 \wedge \langle \alpha_1, \beta_1 \rangle \in R_{\alpha_0}.$$

Let
$$\mathcal{O} = \{\alpha \in \mathcal{O}_+ : \neg \exists f : \mathbb{N} \to \mathcal{O}_+ \, (f(0) = \alpha \wedge \forall n \, f(n+1) <_\mathcal{O} f(n))\},$$

i.e. denote the subclass of indices α such that $<_\mathcal{O}$ is a well-ordering on the initial segment before α. Given any $\alpha \in \mathcal{O}$, we say its *order type*[7] is precisely the order type of this initial segment

$$\text{otyp}(\alpha) = \text{otyp}(\{\beta \in \mathcal{O}_+ : \beta <_\mathcal{O} \alpha\}).$$

Definition (H-sets)**.** The *Turing jump* of a set $X \subseteq \mathbb{N}$ is given by

$$\text{TJ}(X) = \{\langle e, m \rangle : \pi(e, m, X)\}.$$

The following formula identifies when Y represents the iteration of Turing jumps along some ordering R_{α_0} below α_1:

$$H(\alpha, Y) \quad :\Leftrightarrow \quad \alpha \in \mathcal{O}_+ \wedge Y = (Y)_{<_\mathcal{O}\alpha} \wedge \forall \beta <_\mathcal{O} \alpha \, (Y)_\beta = \text{TJ}\Big((Y)_{<_\mathcal{O}\beta}\Big),$$

where the truncation of Y before an ordinal notation α is

$$(Y)_{<_\mathcal{O}\alpha} = \Big\{ \langle \beta, m \rangle : \beta <_\mathcal{O} \alpha \wedge m \in (Y)_\beta \Big\}.$$

By theorem VIII.3.15 in [7], ATR$_0$ proves $\forall \alpha \in \mathcal{O} \, \exists Y \, H(\alpha, Y)$. We can thus make use of the following property, which is lemma VIII.3.14 in [7]:

Proposition 4.1 (ACA$_0$)**.** *Given* $\alpha, \beta \in \mathcal{O}$ *and some* Y *such that* $H(\alpha, Y)$*, then the order types* $\text{otyp}(\alpha)$ *and* $\text{otyp}(\beta)$ *are comparable, and the comparison map is* $\leq_T Y$*.*

Corollary 4.2 (ATR$_0$)**.** *There exists* $\text{cmp} \in [\mathcal{O}, S_2]$ *such that*

$$\{\{\text{cmp}\}(\alpha)\}(\beta) = 1 \quad \Leftrightarrow \quad \beta \in \mathcal{O} \wedge \text{otyp}(\beta) \leq \text{otyp}(\alpha).$$

Proof. Given $\alpha \in \mathcal{O}$, fix Y to be a set such that $H(\alpha, Y)$. We let $\{\{\text{cmp}\}(\alpha)\}(\beta)$ be 1 if $\beta \in \mathcal{O}_+$ and there exists $e \in \text{Rec}^Y$ such that R_e^Y is an order-isomorphism between $\{\gamma \in \mathcal{O}_+ : \gamma <_\mathcal{O} \beta\}$ and an initial segment of $\{\gamma \in \mathcal{O}_+ : \gamma <_\mathcal{O} \alpha\}$. Let $\{\{\text{cmp}\}(\alpha)\}(\beta)$ be 0 otherwise. Notice that this statement is Σ_1^1.

Obviously, suppose that $\{\{\text{cmp}\}(\alpha)\}(\beta) = 1$, then $\beta \in \mathcal{O}$ because any strictly decreasing sequence below β will correspond to a strictly decreasing sequence below α through R_e^Y. Conversely, suppose that indeed $\beta \in \mathcal{O}$ and $\text{otyp}(\beta) \leq \text{otyp}(\alpha)$, then the proposition above ensures that the comparison function is Y-recursive, so $\{\{\text{cmp}\}(\alpha)\}(\beta)$ must be 1. $\qquad\square$

---

[7] In our setup, we will not define concrete representations for order types of well-orderings. Instead, we will treat statements $\text{otyp}(X) = \text{otyp}(Y)$ and $\text{otyp}(X) \leq \text{otyp}(Y)$ as abbreviations for the assertions that corresponding order-preserving injections/bijectons exist respectively.

4.2 Code conversion

Denote
$$\mathcal{H} = \left\{ a : a_0 \in \mathcal{O} \wedge \exists Y \left(H(a_0, Y) \wedge a_1 \in \mathrm{Rec}^Y \right) \right\}.$$

We say that a class $X \subseteq \mathbb{N}$ is *hyperarithmetic*, denoted $X \leq_H \varnothing$, if there exists $a \in \mathcal{H}$ and $Y \subseteq \mathbb{N}$ such that $H(a_0, Y)$ and $X = R_{a_1}^Y$. We say that $a = \langle a_0, a_1 \rangle$ is a *hyperarithmetic index* of X.

The aim of this section is to show that one can convert between the hyperarithmetic index of a set and the Σ_1^1 function code of its characteristic function. Namely, we will prove the following main theorem:

Theorem 4.3 (ATR$_0$). *There exists* $\mathrm{hyp} \in [S_2, \mathcal{H}]$ *such that*

- *for each* $e \in S_2$, $\{\mathrm{hyp}\}(e)$ *is a hyperarithmetic index of* $\{n : \{e\}(n) = 1\}$;

- *for* $d, e \in S_2$, $\{\mathrm{hyp}\}(d) = \{\mathrm{hyp}\}(e)$ *if and only if* $d \equiv e$.

Proof. To begin, we will replicate some constructions in the proof of theorem VIII.3.19 in [7] on the set $\{n : \{e\}(n) = 1\}$: observe that the formula

$$\{e\}(n) = 1 \quad \Leftrightarrow \quad \{e\}(n) \neq 0 \quad \Leftrightarrow \quad \forall X \, \neg \pi(e_0, e_1, n, 0, X)$$

is Π_1^1, for any fixed parameter $e \in S_2$. By Kleene normal form theorem (theorem V.1.4 in [7]), there exists a Σ_0^0-formula $\theta(e, n, \tau)$ such that ACA$_0$ proves

$$\forall n \left(\{e\}(n) = 1 \leftrightarrow \forall f \, \exists k \, \neg \theta(e, n, f[k]) \right)$$

where f ranges over second-order objects representing functions $\mathbb{N} \to \mathbb{N}$, and $f[k] = \langle f(0), \dots, f(k-1) \rangle \in \mathbb{N}$. We then consider recursive trees

$$T_n^{(e)} = \{ \tau \in \mathrm{Seq} : \forall k \leq \mathrm{len}(\tau) \, \theta(e, n, \tau[k]) \}$$

and let $\mathrm{KB}_n^{(e)} = \mathrm{KB}\left(T_n^{(e)} \right)$ be the Kleene-Brouwer ordering of $T_n^{(e)}$. Following the discussions in section V.1 in [7], we have

$$
\begin{aligned}
\{e\}(n) = 1 \quad &\Leftrightarrow \quad \forall f \, \exists k \, \neg \theta(e, n, f[k]) \\
&\Leftrightarrow \quad T_n^{(e)} \text{ has no path} \\
&\Leftrightarrow \quad \mathrm{KB}_n^{(e)} \text{ is a recursive well-ordering.}
\end{aligned}
$$

Now, as mentioned in subsection 4.1, we can assume that $\forall \alpha \in \mathcal{O} \, \exists Y \, H(\alpha, Y)$. Then Simpson's argument applies to justify that the collection of well-orderings $\mathrm{KB}_n^{(e)}$ (for all $n \in \mathbb{N}$, given a fixed e) is bounded in $\mathcal{O}$. Let $\alpha \in \mathcal{O}$ be such a bound and $\Omega \subseteq \mathbb{N}$ be such that $H(\alpha, \Omega)$. By lemma VIII.3.14 in [7], the order types of any initial segment of $\mathrm{KB}_n^{(e)}$ and any $\beta \leq_\mathcal{O} \alpha$ will all be comparable via comparison maps $\leq_T \Omega$, which means for any such well-orderings A, B, a comparison statement $\mathrm{otyp}(A) \leq \mathrm{otyp}(B)$ will be an arithmetic formulae "there exists $a \in \mathbb{N}$ such that R_a^Ω is an order isomorphism between A and an initial segment of B".

Therefore, by Δ_1^1-comprehension,

$$\Gamma = \left\{ \beta \leq_\mathcal{O} \alpha : \forall n \left(\{e\}(n) = 1 \to \mathrm{otyp}\left(\mathrm{KB}_n^{(e)} \right) \leq \mathrm{otyp}(\beta) \right) \right\}$$

is a set and we can find $\lambda = \min \Gamma$. It is easy to also observe that for any $\gamma <_{\mathcal{O}} \lambda$, there must exist $n \in \mathbb{N}$ such that $\{e\}(n) = 1$ and $\mathrm{otyp}(\gamma) < \mathrm{otyp}\left(\mathrm{KB}_n^{(e)}\right)$. In fact, these two characteristics determine λ uniquely up to order isomorphism, i.e. we have the lemma that

Lemma. *Suppose that $\rho \in \mathcal{O}_+$ satisfies both $\mathrm{otyp}\left(\mathrm{KB}_n^{(e)}\right) \le \mathrm{otyp}(\rho)$ for any n such that $\{e\}(n) = 1$, and also for any $\gamma <_{\mathcal{O}} \rho$, there must exist $n \in \mathbb{N}$ such that $\{e\}(n) = 1$ and $\mathrm{otyp}(\gamma) < \mathrm{otyp}\left(\mathrm{KB}_n^{(e)}\right)$, then $\mathrm{otyp}(\rho) = \mathrm{otyp}(\lambda)$.*

Proof. Firstly, assume that $\rho \notin \mathcal{O}$, then there exists a strictly decreasing sequence $f : \mathbb{N} \to \mathcal{O}_+$ such that $f(0) = \rho$. We have $f(1) <_{\mathcal{O}} \rho$, hence we can consider some order-preserving map $g : \{\gamma : \gamma <_{\mathcal{O}} f(1)\} \to \mathrm{KB}_n^{(e)}$. Now $m \mapsto g(f(m+2))$ is a descending sequence in $\mathrm{KB}_n^{(e)}$, a contradiction.

Therefore, we must have $\rho \in \mathcal{O}$. We then know that the order types of ρ and λ are comparable, but it is easy to verify that we can have neither $\mathrm{otyp}(\rho) > \mathrm{otyp}(\lambda)$ nor $\mathrm{otyp}(\rho) < \mathrm{otyp}(\lambda)$. $\qquad \square$

Finally, given that $\lambda \in \mathcal{O}$, we can remove the ambiguity around order-isomorphic codes by requiring that λ is minimal among all order-isomorphic elements in $\mathcal{O}$, with respect to the natural ordering of $\mathbb{N}$. Such a minimum exists by Σ_0^0-induction because, using some $Y \subseteq \mathbb{N}$ satisfying $H(\lambda, Y)$ as a parameter, the set of elements in $\mathcal{O}$ order-isomorphic to λ is an arithmetic set. The rest of Simpson's argument in VIII.3.19 applies to show that if $H(\lambda + \omega, Y)$, then $\{n : \{e\}(n) = 1\} \le_T Y$.

Therefore, we can stipulate $\{\mathrm{hyp}\}(e)$ to compute some $a \in \mathbb{N}$ such that there exists $\alpha \in \mathcal{O}_+$ and $Y \subseteq \mathbb{N}$ and we have

- $H(a_0, Y)$ and[8] $R_{a_0} = R_{\alpha + \omega}$;

- for any $n \in \mathbb{N}$, either $\{e\}(n) = 0$ or $\mathrm{otyp}\left(\mathrm{KB}_n^{(e)}\right) \le \mathrm{otyp}(\alpha)$;

- for any $\beta <_{\mathcal{O}} \alpha$, there exists $n \in \mathbb{N}$ such that $\{e\}(n) = 1$ and $\mathrm{otyp}(\beta) < \mathrm{otyp}\left(\mathrm{KB}_n^{(e)}\right)$;

- for any $\gamma \in \mathcal{O}_+$, if γ is order-isomorphic to a_0, then $\gamma \ge a_0$;

- $a_1 \in \mathrm{Rec}^Y$ and for any $n \in \mathbb{N}$, $\{e\}(n) = 1 \vee n \notin R_{a_1}^Y$ and $\{e\}(n) = 0 \vee n \in R_{a_1}^Y$;

- for any $b \in \mathrm{Rec}^Y$, if $n \in R_{a_1}^Y \leftrightarrow n \in R_b^Y$ holds for any $n \in \mathbb{N}$, then $b \ge a_1$.

This determines a uniquely and is Σ_1^1. Since our analysis above shows that such a must exist for any $e \in S_2$, we have $S_2 \subseteq \mathrm{dom}(\mathrm{hyp})$.

To verify that $\{\mathrm{hyp}\}(d) = \{\mathrm{hyp}\}(e)$ if and only if $d \equiv e$, simply observe that $\{\mathrm{hyp}\}(e)$ is uniquely determined by the set $\{n : \{e\}(n) = 1\}$. $\qquad \square$

A (rough) inverse of the conversion is very easy to compute:

---

[8]Here we fix $R_{\alpha + \omega}$ to mean the following specific recursive well-ordering: simply order the set $\{2\beta + 2 : \beta <_{\mathcal{O}} \alpha\}$ by
$$2\beta_1 + 2 < 2\beta_2 + 2 \quad :\Leftrightarrow \quad \beta_1 <_{\mathcal{O}} \beta_2,$$
and append the sequence $1, 3, 5, 7, \ldots, 0$. Then the initial segment before 0 has order type $\alpha + \omega$.

Proposition 4.4 (ACA$_0$). *There exists* inv $\in [\mathcal{H}, S_2]$ *such that, for any $a \in \mathcal{H}$ and any $Y \subseteq \mathbb{N}$ satisfying $H(a_0, Y)$,*

$$\{\{\mathrm{inv}\}(a)\}(n) = 1 \quad \Leftrightarrow \quad n \in R^Y_{a_1}.$$

Proof. This is simply because the formula

$$\{\{\mathrm{inv}\}(a)\}(n) = i \quad :\Leftrightarrow \quad \exists Y\,(H(a_0, Y) \wedge ((\varphi(a, n, Y) \wedge i = 1) \vee (\neg\varphi(a, n, Y) \wedge i = 0)))$$

is Σ^1_1, where $\varphi(a, n, Y) = a_1 \in \mathrm{Rec}^Y \wedge n \in R^Y_{a_1}$. $\qquad\square$

4.3 Realisability of a global well-ordering

The Σ^1_1 function cmp defined in Corollary 4.2 is a well-founded preorder on $\mathcal{O}$. This easily induce a relation[9] cmp$^* \in [\mathcal{H}, S_2]$ such that $\{\{\mathrm{cmp}^*\}(a)\}(b) = 1$ if and only if both

(i) $\{\{\mathrm{cmp}\}(a_0)\}(b_0) = 1 \wedge (\{\{\mathrm{cmp}\}(b_0)\}(a_0) = 0 \vee (\{\{\mathrm{cmp}\}(b_0)\}(a_0) = 1 \wedge b < a))$,

(ii) $b \in \mathcal{H}$, i.e. $\exists Y\,\big(H(b_0, Y) \wedge b_1 \in \mathrm{Rec}^Y\big)$.

Observe that our definition of cmp ensures that $\{\{\mathrm{cmp}\}(a_0)\}(b_0) = 1$ implies $b_0 \in \mathcal{O}$ and hence the corresponding H-set exists, thus the negation of the conjunction (i) $\wedge$ (ii) above is

$$\neg\text{(i)} \vee \big(\text{(i)} \wedge \exists Y\,\big(H(b_0, Y) \wedge b_1 \notin \mathrm{Rec}^Y\big)\big).$$

This is Σ^1_1 and we can set $\{\{\mathrm{cmp}^*\}(a)\}(b) = 0$ when this holds, concluding that the Σ^1_1 function cmp* is well-defined.

Proposition 4.5 (ATR$_0$). *The relation $\prec$ on $\mathcal{H}$ given by*

$$b \prec a \quad :\Leftrightarrow \quad \{\{\mathrm{cmp}^*\}(a)\}(b) = 1$$

is a well-ordering.

Proof. $\prec$ is a linear ordering simply because any two elements in $\mathcal{O}$ are comparable.

Suppose it is not a well-ordering, i.e. we have a second-order object $f : \mathbb{N} \to S^*_2$ that is a strictly decreasing function. Then $f_0 : \mathbb{N} \to \mathcal{O}$ given by $f_0(0) = f(n)_0$ satisfies

$$\mathrm{otyp}(f_0(0)) \geq \mathrm{otyp}(f_0(1)) \geq \cdots.$$

In other words, we have $\mathrm{otyp}(f_0(0)) \geq \mathrm{otyp}(f_0(n))$ for any $n \in \mathbb{N}$, i.e. there exists a unique $\gamma_n \leq_{\mathcal{O}} f_0(0)$ such that $\mathrm{otyp}(f_0(n)) = \mathrm{otyp}(\gamma_n)$. Now, fix $\Omega \subseteq \mathbb{N}$ such that $H(f_0(0), \Omega)$, then all the comparison statements will be arithmetic with parameter Ω. Thus, by arithmetic comprehension, there exists a function $f' : n \mapsto \gamma_n$ and we immediately have $f_0(0) = f'(0) \geq_{\mathcal{O}} f'(1) \geq_{\mathcal{O}} \cdots$. Since $f_0(0) \in \mathcal{O}$, f' must stabilise after some N, implying that $\mathrm{otyp}(f_0(m)) = \mathrm{otyp}(f_0(n))$ for any $m, n > N$.

Finally, for any $m > N$, we have $f(m) \succ f(m+1)$ yet $\mathrm{otyp}(f(m)_0) = \mathrm{otyp}(f(m+1)_0)$, thus we must have $f(m) > f(m+1)$ by the definition of cmp*. Thus, f restricted to $\{m : m > N\}$ is a strictly decreasing sequence of natural numbers. This contradicts Σ^0_0-induction. Therefore, such f cannot exist and $\prec$ is indeed a well-ordering. $\quad\square$

---

[9] We used the most natural definitions when defining cmp in Corollary 4.2 and when defining cmp* here. The consequence is that cmp is a non-strict (i.e. reflective) ordering on $\mathcal{O}$ but cmp* is a strict (i.e. asymmetric) ordering on $\mathcal{H}$.

With the set-up so far, we will show that ATR can realise the axioms (W1), (W3) in GWO. As later parts of this section shall show, the entire extension CM + GWO is unfortunately too strong and has to be dealt with in the system BI instead.

To interpret axioms (W1), (W3) in ATR, we shall use the same realisability conditions as in subsection 2.3, but fix the interpretation of $\prec$ as

$$\{r\}(e) = \{\{\mathrm{cmp}^*\}(\{\mathrm{hyp}\}(e_1))\}(\{\mathrm{hyp}\}(e_0)).$$

As stipulated in Theorem 4.3, this definition indeed satisfies the requirement above that

$$\forall e, e', f, f' \in S_2 \left(e \equiv f \wedge e' \equiv f' \to \{r\}(\langle e, e' \rangle) = \{r\}(\langle f, f' \rangle) \right).$$

Now, first observe that the axiom set (W1) stating that $\prec$ is a linear ordering is simply interpreted as propositional compositions of atomic formulae of the form $\{\mathrm{hyp}\}(e) \prec \{\mathrm{hyp}\}(f)$. Since this is indeed a linear ordering modulo extensional equality on S_2, these axioms are realised by any Σ_1^1 function with appropriate domain and range. We also have

Proposition 4.6. *The axiom of countable initial segments* (W3) *is realised:*

$$\forall X \, \exists Z \, \forall Y \left(Y \prec X \to \exists n \, Y = (Z)_n \right).$$

Proof. Consider any $x \in S_2$. We make use of the fact that $\{\mathrm{cmp}^*\}(\{\mathrm{hyp}\}(x)) \in S_2$ is a Σ_1^1 function defining the initial segment $W = \{y \in \mathcal{H} : y \prec \{\mathrm{hyp}\}(x)\}$. Therefore, the desired Z can simply be constructed as the code $z \in S_2$ given by

$$\{z\}(\langle m, n \rangle) = \begin{cases} \{\{\mathrm{inv}\}(m)\}(n) & \text{if } \{\{\mathrm{cmp}^*\}(\{\mathrm{hyp}\}(x))\}(m) = 1, \\ 0 & \text{if } \{\{\mathrm{cmp}^*\}(\{\mathrm{hyp}\}(x))\}(m) = 0. \end{cases}$$

Our construction ensures that if any $y \in S_2$ satisfies $\{\mathrm{hyp}\}(y) \prec \{\mathrm{hyp}\}(x)$, then $(Z)_{\{\mathrm{hyp}\}(y)} = Y$. Therefore, the $\forall Y \left(Y \prec X \to \exists n \, Y = (Z)_n \right)$ is simply realised by

$$\lambda y.\lambda d. \left\langle \{\mathrm{hyp}\}(y), * \right\rangle,$$

where $*$ is any function with appropriate domain and range. $\qquad \square$

4.4   Unrestricted transfinite induction

Finally, we work on the axiom scheme of unrestricted transfinite induction (W2):

$$\forall X \left(\forall Y \left(Y \prec X \to \varphi(Y) \right) \to \varphi(X) \right) \to \forall X \, \varphi(X)$$

for any third-order formula $\varphi(X)$.

To realise this, we move to the stronger second-order theory BI, that is, the axioms in ACA_0 plus the scheme of transfinite induction along any countable well-ordering:

$$\forall X \, \forall <_X \left(\mathrm{WO}(X, <_X) \to \forall x \in X \left(\forall y <_X x \, \varphi(y) \to \varphi(x) \right) \to \forall x \in X \, \varphi(x) \right)$$

for any formula $\varphi(x)$, where $\mathrm{WO}(X, <_X)$ denotes the Π_1^1-formula asserting that $<_X$ is a linear ordering on X and there exists no strictly decreasing function $f : \mathbb{N} \to X$. This is a further extension of the classical theories we worked with above due to corollary VII.2.19 in Simpson [7]:

Fact. BI *implies* ATR.

By working in the theory BI, we additionally know how to iterate Σ_1^1 functions along any well-ordering:

Theorem 4.7 (BI). *Given $x, y \in S_2$ and write*

$$X = \{n : \{x\}(n) = 1\}, \qquad <_X = \{n : \{y\}(n) = 1\},$$

which exist as hyperarithmetic sets. Suppose that $\mathrm{WO}(X, <_X)$. Fix some formula $\varphi(a, z)$ and suppose f is a Σ_1^1 function code such that for any $a \in X$ and $e \in \mathbb{N}$, if for any $b \in X$, $b <_X a$, we have $b \in \mathrm{dom}(e)$ and $\varphi(b, \{e\}(b))$, then $\langle a, e \rangle \in \mathrm{dom}(f)$ and $\varphi(a, \{f\}(\langle a, e \rangle))$. Then there is a Σ_1^1 function code g such that

- $X \subseteq \mathrm{dom}(g)$,

- $\varphi(a, \{g\}(a))$ *for any $a \in X$,*

- $\{g\}(a) = \{f\}(\langle a, g \rangle)$ *for any $a \in X$.*

Additionally, g can be computed from f, x and y uniformly by a Σ_1^1 coded function.

Proof. Let d be a Σ_1^1 code that computes $\lambda h.\lambda a.\{h\}(\langle a, h \rangle)$. We now simply compute a code for the following formula

$$
\begin{aligned}
\{h\}(a) = z \quad :\Leftrightarrow \quad &\{x\}(a_0) = 1 \\
&\wedge \forall b \left(\{y\}(\langle b, a_0 \rangle) = 1 \rightarrow \{a_1\}(\langle b, a_1 \rangle) = \{f\}(\langle b, \{d\}(a_1) \rangle) \right) \\
&\wedge z = \{f\}(\langle a_0, \{d\}(a_1) \rangle),
\end{aligned}
$$

and we let $g = \{d\}(h)$. We have yet to show that the formula for h on top defines a function indeed, but it will suffice if we can show the following formula

$$\psi(a) := a \in \mathrm{dom}(g) \wedge \{g\}(a) = \{f\}(\langle a, g \rangle) \wedge \varphi(a, \{g\}(a))$$

for all $a \in X$. Notice that suppose we have $\psi(b)$ for all $b <_X a$, then for each b we have $\langle b, h \rangle \in \mathrm{dom}(h)$ and indeed $\{h\}(\langle b, h \rangle) = \{f\}(\langle b, g \rangle)$ satisfies $\varphi(b, \{h\}(\langle b, h \rangle))$. Thus $\langle a, g \rangle \in \mathrm{dom}(f)$ and $\{h\}(\langle a, h \rangle) = z$ is satisfied by the unique z such that $z = \{f\}(\langle a, g \rangle)$, which also satisfies $\varphi(a, z)$ by our assumption. Therefore, using BI on ψ, we can immediately conclude $\psi(a)$ for all $a \in X$.

Finally, the fact that g can be computed using a Σ_1^1 function code simply follows from Proposition 2.3, treating the inputs as parameters for the function h. $\qquad \square$

Therefore, assuming BI, we can finally show that the realisability conditions introduced above satisfy transfinite induction for any formula $\varphi(X)$ on the well-ordering $\prec$:

Lemma 4.8 (Σ_1^1-AC). *For any $d \in \mathbb{N}$ and $e, f \in S_2$ such that $e \equiv f$, if $d \Vdash \varphi(e)$ for some formula $\varphi(X)$, then likewise $d \Vdash \varphi(f)$.*

Specifically, assuming ATR, *then $d \Vdash \varphi(e)$ if and only if $d \Vdash \varphi(\{\mathrm{inv}\}(\{\mathrm{hyp}\}(e)))$.*

Proof. Observe that when φ is atomic, d is irrelevant and this holds simply because our realisability conditions realise the Leibniz law. Thus, this holds for any formula $\varphi(X)$ by simple induction on the complexity of φ. $\qquad \square$

Proposition 4.9 (BI)**.** *The axiom of unrestricted transfinite induction* (W2) *is realised:*

$$\forall X \, (\forall Y \, (Y \prec X \to \varphi(Y)) \to \varphi(X)) \to \forall X \, \varphi(X).$$

Proof. For any $f \Vdash \forall X \, (\forall Y \, (Y \prec X \to \varphi(Y)) \to \varphi(X))$ and any $x \in S_2$, we need to compute some d such that $d \Vdash \varphi(x)$. Again use that $\{\mathrm{cmp}^*\}(\{\mathrm{hyp}\}(x)) \in S_2$ is a Σ_1^1 function and we can consider the set

$$W = \{y \in \mathcal{H} : \{\{\mathrm{cmp}^*\}(\{\mathrm{hyp}\}(x))\}(y) = 1 \wedge \{\mathrm{hyp}\}(\{\mathrm{inv}\}(y)) = y\}.$$

Notice that for any $y \in S_2$, we have $\{\mathrm{hyp}\}(\{\mathrm{inv}\}(\{\mathrm{hyp}\}(y))) = \{\mathrm{hyp}\}(y)$ by stipulation in Theorem 4.3, thus $\{\mathrm{hyp}\}(y) \in W$ as long as $\{\mathrm{hyp}\}(y) \prec \{\mathrm{hyp}\}(x)$.

Here we have $\mathrm{WO}(W, \prec)$ by Proposition 4.5, thus we can iterate a Σ_1^1 function on W. We will iterate the function g' given by

$$\lambda a.\{\{f\}(\{\mathrm{inv}\}(a_0))\}(\{d\}(a_1))$$

where d computes $\lambda e.\lambda z.\lambda h.\{e\}(\{\mathrm{hyp}\}(z))$. Observe that for $y \in W$ and $e \in \mathbb{N}$, if for any $z \in W$, $z \prec y$, we have $\{e\}(z) \Vdash \varphi(\{\mathrm{inv}\}(z))$, then

$$\{d\}(e) \Vdash \forall Z \, (Z \prec \{\mathrm{inv}\}(y) \to \varphi(Z))$$

by Lemma 4.8 and hence $\{g'\}(\langle y, e \rangle) \Vdash \varphi(\{\mathrm{inv}\}(y))$. Therefore, iterating g' on W will produce a Σ_1^1 function code g such that for any $y \in W$, $\{g\}(y) \Vdash \varphi(\{\mathrm{inv}\}(y))$. Then $\{g'\}(\langle\{\mathrm{hyp}\}(x), g\rangle) \Vdash \varphi(x)$ by Lemma 4.8 again and is precise the number we need to compute. $\square$

We have thus shown that in BI, our realisability conditions interpret all intuitionistic consequences of $\mathrm{CM} + \mathrm{GWO}$.

4.5   An ordinal analysis of $\mathrm{CM} + \mathrm{GWO}$

We say that a formula $\varphi(X, x)$ in the language of CM is a *strictly positive first-order formula*, possibly with other free variables, if it is built up according the following rules:

- φ can contain any atomic formulae in the language of PA, as well as atomic formulae of the form $t \in_1 X$ where t is an arbitrary first-order term;

- φ can contain the propositional constant "bottom": $\bot$;

- φ can contain connectives $\wedge$, $\vee$, $\to$, with the restriction that X cannot occur on the left of the connective $\to$;

- φ can contain quantifiers $\forall y$ and $\exists y$ where y is any first-order variable.

Let $\theta(x)$ be any formula, then we let $\varphi(\theta, x)$ denote the result of substituting every occurrence of $t \in_1 X$ inside $\varphi(X, x)$ by $\theta(t)$, with appropriate renaming so that any other free variable in $\theta(x)$ is not bound by any quantifiers in φ.

We shall prove the following:

Theorem 4.10. *For any strictly positive first-order formula $\varphi(X, x)$, there exists $\theta_\varphi(x)$ such that*

$$\text{CM} + \text{GWO} \vdash \forall x \left(\varphi(\theta_\varphi, x) \to \theta_\varphi(x) \right),$$

and also for any formula $\eta(x)$

$$\text{CM} + \text{GWO} \vdash \forall x \left(\varphi(\eta, x) \to \eta(x) \right) \to \forall x \left(\theta_\varphi(x) \to \eta(x) \right).$$

This will involve a few lemmata:

Lemma 4.11 (Positivity). *For any formulae $\theta(x), \eta(x)$,*

$$\text{CM} \vdash \forall x \left(\theta(x) \to \eta(x) \right) \to \forall x \left(\varphi(\theta, x) \to \varphi(\eta, x) \right).$$

This trivially implies $X \subseteq Y \to \forall x \left(\varphi(X, x) \to \varphi(Y, x) \right)$ as well.

Proof. This follows from an easy induction on the complexity of $\varphi(X, x)$. □

Lemma 4.12. *For any formula $\theta(x)$,*

$$\text{CM} \vdash \forall x \left(\varphi(\theta, x) \to \exists X \left(\forall z \in_1 X \, \theta(z) \wedge \varphi(X, x) \right) \right)$$

Proof. We show this by induction on the complexity of $\varphi(X, x)$:

Firstly, if $\varphi(X, x)$ is an atomic formula not of the form $t \in_1 X$, then $\varphi(\theta, x) \to \varphi(\varnothing, x)$ trivially; if $\varphi(X, x)$ is $t \in_1 X$, then $\varphi(\theta, x) \to \forall z \in_1 \{t\} \, \theta(z) \wedge \varphi(\{t\}, x)$.

The case where φ is $\perp$ is vacuously true.

Now, suppose that φ is of the form $\psi \wedge \chi$ and $\varphi(\theta, x)$ holds, then by the inductive hypothesis we have X and Y such that

$$\forall z \in_1 X \cup Y \, \theta(z) \wedge \psi(X, x) \wedge \chi(Y, x).$$

It follows from Lemma 4.11 that we also have $\psi(X \cup Y, x) \wedge \chi(X \cup Y, x)$.

Suppose φ is of the form $\psi \to \chi$ and $\varphi(\theta, x)$ holds, we know from stipulation that $\psi(x)$ must not contain X. By the inductive hypothesis, we have

$$\psi(x) \to \exists X \left(\forall z \in_1 X \, \theta(z) \wedge \chi(X, x) \right).$$

Since $\psi(x)$ is arithmetic, hence decidable, we can take Y to be X above if $\psi(x)$ holds and $\varnothing$ otherwise. Then we always have $\forall z \in_1 Y \, \theta(z)$, and we also have $\psi(x) \to \chi(Y, x)$.

Suppose φ is of the form $\forall y \, \psi$ and $\varphi(\theta, x)$ holds, then by the inductive hypothesis

$$\forall y \, \exists X \left(\forall z \in_1 X \, \theta(z) \wedge \psi(X, x, y) \right).$$

By dependent choice, we can find Z such that each $(Z)_y$ satisfies $\forall z \in_1 (Z)_y \, \theta(z) \wedge \psi\left((Z)_y, x, y \right)$ and take $Y = \bigcup_y (Z)_y$. Then $\forall z \in_1 Y \, \theta(z)$ and by Lemma 4.11, we have $\forall y \, \psi(Y, x, y)$ as desired.

Finally, the cases where φ is of the form $\psi \vee \chi$ or $\exists y \, \psi$ are trivial. □

Proof of Theorem 4.10. Let $\psi(X, Z, S)$ assert that both of the following holds:

- $\forall Y \preceq X \, \exists n \, Y = (Z)_n$;
- $\forall n \, ((Z)_n \preceq X \to \forall x \, (x \in_1 (S)_n \leftrightarrow \varphi(\bigcup \{(S)_m : (Z)_m \prec (Z)_n\}, x)))$.

By transfinite induction on all second-order objects, it is easy to show that for each X, a pair Z, S exists and is unique in the following sense:

$$\psi(X, Z, S) \wedge \psi(X, Z', S') \to \forall n, m \, ((Z)_n \preceq X \wedge (Z)_n = (Z')_m \to (S)_n = (S')_m).$$

Now, we let $\theta_\varphi(x) := \exists X, Z, S \, (\psi(X, Z, S) \wedge \exists n \, (X = (Z)_n \wedge x \in_1 (S)_n))$. We will verify that CM + GWO proves the desired properties of $\theta_\varphi(x)$:

Firstly, suppose that $\varphi(\theta_\varphi, x)$ holds for some x. By Lemma 4.12, we can find A satisfying

$$\forall x \in_1 A \, \theta_\varphi(x) \wedge \varphi(A, x).$$

Using dependent choice, the former conjunct will implies the existence of X, Z, S such that

$$\forall x \in_1 A \, (\psi((X)_x, (Z)_x, (S)_x) \wedge \exists n \, ((X)_x = ((Z)_x)_n \wedge x \in_1 ((S)_x)_n)).$$

Using the fact that the $\prec$-initial segment before each $(X)_x$ is countable and dependent choice again, we can easily find Z such that

$$\forall x \in_1 A \, \forall Y \, (Y \preceq (X)_x \to \exists n \, Y = (Z)_n),$$

so taking $Y = \{n : \neg n \in_1 (Z)_n\}$ and we must have $\forall x \in_1 A \, (X)_x \prec Y$. Let Z^*, S^* be such that $\psi(Y, Z^*, S^*)$ holds, then it follows that

$$A \subseteq \bigcup \{(S^*)_m : (Z^*)_m \prec Y\}.$$

Immediately $\varphi(A, x)$ and Lemma 4.11 will imply $\theta_\varphi(x)$.

Secondly, suppose that for some formula $\eta(x)$, $\forall x \, (\varphi(\eta, x) \to \eta(x))$ holds. In order to have $\forall x \, (\theta_\varphi(x) \to \eta(x))$, it suffices to show by transfinite induction that

$$\forall X, Z, S \, (\psi(X, Z, S) \to \forall n \, (X = (Z)_n \to \forall x \in_1 (S)_n \, \eta(x))).$$

This is essentially because, when $X = (Z)_n$, the inductive hypothesis will imply

$$\forall x \in_1 \bigcup \{(S)_m : (Z)_m \prec X\} \, \eta(x),$$

thus for any $x \in_1 (S)_n$, i.e. any x satisfying $\varphi(\bigcup \{(S)_m : (Z)_m \prec X\}, x)$, Lemma 4.11 would imply $\varphi(\eta, x)$ and thus $\eta(x)$ as required. $\qquad \square$

Since HA is trivially a subtheory of CM, the theorem implies that CM+GWO interprets the intuitionistic[10] first-order theory of inductive definitions ID_1^i. The proof-theoretic ordinal of ID_1^i is computed in [2] as $|\mathrm{ID}_1^i| = \theta_{\varepsilon_{\Omega+1}}(0) = |\mathrm{BI}|$. Since CM + GWO can be realised in BI, it is an immediate corollary that

Corollary 4.13. CM + GWO $\equiv$ BI. *In particular, we have* $|\mathrm{CM} + \mathrm{GWO}| = |\mathrm{BI}| = \theta_{\varepsilon_{\Omega+1}}(0)$.

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[10] Even though CM $\vdash$ PA, we can only interpret the intuitionistic part here because we are translating the first-order predicate symbol I_φ for the least fixed point of φ into a non-decidable formula $\theta_\varphi(x)$.

5 Indispensability of countable initial segments

In this section, we will point out an interesting contrast between the extension CM + GWO and the more conventional attempts of attaching a global well-ordering to a base theory in classical logic. For example, over Kripke–Platek set theory, one can take Gödel's constructible universe L as a witness of that KP is equiconsistent with "KP + there is a global well-ordering of all sets". As an immediate by-product, the first uncountable ordinal ω_1, if it exists, is regular and hence an admissible ordinal in KP, thus L_{ω_1} will be a model of

KP + there is a global well-ordering whose every initial segment is countable,

rendering this theory also equiconsistent with KP.

Through the ordinal analysis in the sections above, we have shown that adding the global well-ordering axioms GWO to the intuitionistic system CM, instead, results in an increase in the consistency strength. If we likewise consider the fragment CM + (W1) + (W2) where the axiom of countable initial segments (W3) is dropped, then one may suspect that the simpler truncation argument of L_{ω_1} above may be replicated in the intuitionistic context whereas the more complicated construction of a well-orderable "constructible class" naturally fails, so a quick guess would be that CM + (W1) + (W2) inherits the same consistency strength of the full theory CM + GWO, rendering the extra axiom (W3) redundant.

However, we will show that this is not the case, and the axiom (W3) is indeed indispensable in order to achieve the proof-theoretic ordinal $\theta_{\varepsilon_{\Omega+1}}(0)$. We shall do so by providing an interpretation of the extension CM + (W1) + (W2) in the classical theory ATR.

Let us again consider the realisability conditions given in subsection 2.3, where the interpretation of the well-ordering $\prec$ is instead fixed as

$$\{r'\}(\langle e, f\rangle) = \begin{cases} 0 & \text{if } \{\text{hyp}\}(e) < \{\text{hyp}\}(e), \\ 1 & \text{otherwise.} \end{cases}$$

Recall from Theorem 4.3 that for $e, f \in S_2$, $\{\text{hyp}\}(e) = \{\text{hyp}\}(e)$ if and only if $e \equiv f$, thus r' satisfies the extensionality requirement that

$$\forall e, e', f, f' \in S_2 \left(e \equiv f \wedge e' \equiv f' \to \{r\}(\langle e, e'\rangle) = \{r\}(\langle f, f'\rangle)\right).$$

Once again, we see that the axiom set (W1) stating that $\prec$ is a linear ordering is interpreted as propositional compositions of numerical inequalities in the base theory, thus realised by any Σ_1^1 function with appropriate domain and range. We also have

Proposition 5.1 (ATR). *Using this simple interpretation r', the axiom of unrestricted transfinite induction (W2) is realised:*

$$\forall X \left(\forall Y \left(Y \prec X \to \varphi(Y)\right) \to \varphi(X)\right) \to \forall X\, \varphi(X).$$

Proof. Given any $f \Vdash \forall X \left(\forall Y \left(Y \prec X \to \varphi(Y)\right) \to \varphi(X)\right)$, the traditional recursion theorem for partial combinatory algebras allows us to compute some Σ_1^1 partial function g such that

$$\{g\}(n) = \{\{f\}(\{\text{inv}\}(n))\}(\lambda y.\lambda e.\{g\}(\{\text{hyp}\}(y))).$$

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We can show by induction on $\mathbb{N}$ that the following holds for every $x \in S_2$:

$$\forall y \in S_2 \left(\{\text{hyp}\}(y) \leq \{\text{hyp}\}(x) \to \{\text{hyp}\}(y) \in \text{dom}(g) \wedge \{g\}(\{\text{hyp}\}(y)) \Vdash \varphi(y)\right).$$

Here, the inductive hypothesis implies

$$\lambda y.\lambda e.\{g\}(\{\text{hyp}\}(y)) \Vdash \forall Y \left(Y \prec x \to \varphi(Y)\right),$$

thus the construction of g ensures that $\{\text{hyp}\}(x) \in \text{dom}(g)$ and, by Lemma 4.8, $\{g\}(\{\text{hyp}\}(x)) \Vdash \varphi(x)$ as well. Hence, by induction, this means that

$$\lambda x.\{g\}(\{\text{hyp}\}(x)) \Vdash \forall X \, \varphi(X). \qquad \square$$

Compare this with the results we obtained in subsection 4.3, we observe that both the fragments CM + (W1) + (W2) and CM + (W1) + (W3) are realisable in ATR only — although with different realisers for the well-ordering $\prec$ — but the full extension CM + GWO requires BI. Therefore, indeed all axioms in our extension are necessary for its consistency strength.

6 A concluding remark on impredicativity

In this last section, we will return to focus on the transfinite induction scheme (W2),

$$\forall X \left(\forall Y \left(Y \prec X \to \varphi(Y)\right) \to \varphi(X)\right) \to \forall X \, \varphi(X).$$

As we mentioned when defining the axioms, Weaver did not want this scheme to apply to an arbitrary third-order formula $\varphi(X)$, since it renders the resulting system impredicative. In his own words in [10],

> [transfinite induction] should only be asserted for formulas that do not contain $\prec$, for reasons having to do with the circularity involved in making sense of a relation that is well-ordered with respect to properties that are defined in terms of that relation.

However, while the rejection of impredicative constructions is understandable from the philosophical stance Weaver chose to take, his suggestion quoted here fail to resolve the problem. If we read it in the usual sense, i.e. we forbid the primitive relation symbol $\prec$ from φ but still allow the occurrences of any other free parameters, then given any formula $\varphi(X)$, one can simply apply the transfinite induction scheme to the formula $\tilde{\varphi}(X)$ instead, where every occurrence of $X \prec Y$ has been replaced by $\langle X, Y \rangle \in \mathbf{W}$, given $\mathbf{W}$ being an unused, free third-order parameter. By decidable comprehension, we can later set

$$\mathbf{W} \;\coloneqq\; \{\langle X, Y \rangle : X \prec Y\}$$

to derive transfinite induction for the original formula φ. Therefore, such superficial restriction on φ does not actually restrict the theory in any ways.

However, if we are to read this proposal in a harsher way, by either forbidding the occurrence of any free (third-order) parameter in φ at all, or maintaining that the axiom scheme of decidable comprehension is postulated for the restricted language of CM only and cannot be applied to formulae containing the new symbol $\prec$, then the

utility of the extension will be severely reduced. For example, as we have explored in section 3, the proof of traditional mathematical results like the existence of bases using the global well-ordering axioms will very often involve recursive constructions along the well-ordering and then applications of the induction axiom to prove properties of such constructions. It is unclear how any of those results, including the general version of Hahn–Banach theorem that Weaver himself claimed to have proved, can be established without accessing the initial segment, i.e. second-order sets $Y \prec X$ for a given X in a meaningful way.

Moving away from Weaver's original proposal, it still remains open though whether one can find any interesting theory fragment lying between CM and CM + GWO that both

- has significantly weaker proof-theoretic strength than full CM + GWO; and

- can still prove most of the mathematical results we looked at in section 3.

For example, if we restrict the transfinite induction scheme (W2) to only where the formula $\varphi(X)$ is decidable, then this is equivalent to the single axiom

(W2′) $\forall X \, (\forall Y \, (Y \prec X \to Y \in_2 \mathbf{X}) \to X \in_2 \mathbf{X}) \to \forall X \, X \in_2 \mathbf{X}$

by taking the third-order parameter $\mathbf{X} \coloneqq \{X : \varphi(X)\}$. We can prove that

Proposition 6.1 (ATR)**.** *The axiom of transfinite induction* (W2′) *is realised (using the realiser r defined in subsection 4.3 for the symbol $\prec$).*

Proof. Fix some $h \in S_3$ and suppose that $\forall X \, (\forall Y \, (Y \prec X \to Y \in_2 h) \to X \in_2 h)$ is realised. This implies

$$\forall e \in S_2 \, (\forall f \in S_2 \, (\{\mathrm{hyp}\}(f) \prec \{\mathrm{hyp}\}(e) \to \{h\}(f) = 1) \to \{h\}(e) = 1),$$

and it suffices to show that $\forall e \in S_2 \, \{h\}(e) = 1$ as then any Σ_1^1 function with appropriate domain and range would realise $\forall X \, X \in_2 h$.

Suppose that this is not the case, then there exists some $e \in S_2$ such that $\{h\}(e) = 0$, and it follows that we have some $f \prec \{\mathrm{hyp}\}(e)$ satisfying $\{h\}(\{\mathrm{inv}\}(f)) = 0$. Fix e and consider the set

$$\{f \in \mathbb{N} : \{\{\mathrm{cmp}^*\}(\{\mathrm{hyp}\}(e))\}(f) = 1 \wedge \{h\}(\{\mathrm{inv}\}(f)) = 0\}$$

which exists by Δ_1^1-comprehension since $\{\mathrm{cmp}^*\}(\{\mathrm{hyp}\}(e))$ is a total Σ_1^1 function, then it must be non-empty and thus contains a $\prec$-minimal element. However, this contradicts the assumption above. $\qquad\square$

Combined with results in subsection 4.3, this means that the fragment CM + (W1) + (W2′) + (W3) is entirely realised ATR. However, this fragment is still too weak, as one can observe that the crucial use of the transfinite induction axiom in Proposition 3.2 requires one second-order existential quantifier in the formula φ, and there is no convenient prior justification that the specific φ involved is indeed decidable. It shall be up to further work to determine whether the realisability result Proposition 6.1 can be extended to accommodate formulae of higher complexity.

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