

Nik Weaver's conceptualist ideal and predicative actualisations

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- 1 Nik Weaver's mathematical conceptualism and conceptualist mathematics through third-order arithmetic
- 2 Ordinal analysis, consistency strength and a hyperarithmetic realisability interpretation
- 3 Analysing the predicative global well-ordering principle
- 4 The Feferman–Schütte limit of predicativism and iterations in Weaver's conceptualism

Nik Weaver's arXiv manuscripts

- *Analysis in J_2* (2005a)
- *Mathematical conceptualism* (2005b)
- *Axiomatizing mathematical conceptualism in third order arithmetic* (2009a)
- *The concept of a set* (2009b)
- *Is set theory indispensable?* (2009c)
- *Predicativity beyond Γ_0* (2009d)
- *Kinds of concepts* (2011)
- *Predicative well-ordering* (2018)
- *Hierarchies of Tarskian truth predicates* (2022)

The starting point

As Weaver (2005b) wrote:

Classical set theory presents us with a picture of an incredibly vast universe. [...] Yet virtually all important objects in mainstream mathematics are either countable or separable. This is not because the uncountable/nonseparable case has not yet been sufficiently studied but rather because on mainstream questions it tends to be either pathological or undecidable.

The starting point

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An example from Weaver (2009c):

- Let $V = L^1(\mathbb{R})$ be the Banach space of Lebesgue integrable functions. This is well-known, and separable (i.e. has a countable description);
- Its dual $V' = L^\infty(\mathbb{R})$ is the Banach space of almost everywhere bounded functions. This is still well-known, but nonseparable;
- The next dual V'' — is some obscure space that exists in ZFC.

The countable and uncountable

- Finite processes are physically realisable — this is what the finitists limit themselves to;
- Traditional predicativists like Hermann Weyl “accept only what is already implicit in accepting the natural numbers as given to us.”
- Weaver argues that instead, mathematics should admit all countable processes, which are “*conceptually definite*, meaning that we must be able to form a completely clear mental picture of how the construction would proceed.” (Weaver, 2005b)

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Thus, Weaver (2009d) presents the slogans of *conceptualism* as:

- ① The mathematical universe is a variable entity that can always be enlarged;
- ② Every set must be constructible from logically prior sets;
- ③ Constructions of length ω are legitimate.

Surveyable and definite concepts

Weaver (2011) distinguished between the countable and uncountable infinite as the following:

- A concept is *surveyable* if it is possible, in principle, to exhaustively survey all of the individuals which fall under it;
- A concept is *definite* if any individual definitely either does or does not fall under it.

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Weaver holds a Platonist position only for the surveyable objects, thus:

If we adopt a classical interpretation of truth then we can use classical logic in both the surveyable and definite settings, but we cannot reason about heuristic collections. If we adopt a constructive interpretation of truth then we must use classical logic in the surveyable case, intuitionistic logic in the definite case, and minimal logic in the heuristic case. (Weaver, 2009b)

By “in the surveyable/definite setting”, he meant when we only quantify over surveyable/definite collections, etc.

The system CM

We begin with intuitionistic third-order arithmetic, using $n, m, \dots$ for first-order variables, $X, Y, \dots$ for second-order variables, and $\mathbf{X}, \mathbf{Y}, \dots$ for third-order variables.

We have *extensional* equality for second-order and third-order objects, and number axioms from HA^- .

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The *induction* axiom for any φ :

$$\varphi(0) \wedge \forall n (\varphi(n) \rightarrow \varphi(n+1)) \rightarrow \forall n \varphi(n);$$

The *recursion* axiom (or dependent choice) for any φ :

$$\forall n \forall X \exists Y \varphi(n, X, Y) \rightarrow \forall X \exists Z ((Z)_0 = X \wedge \forall n \varphi(n, (Z)_n, (Z)_{n+1})),$$

where $(Z)_n = \{m \in \mathbb{N} : \langle n, m \rangle \in Z\}$.

The system CM

The *decidable comprehension* axioms

$$\begin{aligned}\forall n (\varphi(n) \vee \neg \varphi(n)) &\rightarrow \exists X \forall n (n \in X \leftrightarrow \varphi(n)), \\ \forall X (\varphi(X) \vee \neg \varphi(X)) &\rightarrow \exists \mathbf{X} \forall X (X \in \mathbf{X} \leftrightarrow \varphi(X));\end{aligned}$$

And the law of excluded middle $\varphi \vee \neg \varphi$ for all arithmetic formulae φ , i.e. formulae that do not contain second- or third-order quantifiers.

Or equivalently, excluded middle for all atomic formulae, together with the principle of *numerical omniscience* (Weaver, 2009a, Theorem 2.2):

$$\forall n (\varphi(n) \vee \psi(n)) \rightarrow (\forall n \varphi(n) \vee \exists n \psi(n)).$$

Mathematics in CM

We can define $\mathbb{R}$ as a third-order object: let $\mathbb{Q}$ be coded in $\mathbb{N}$, then $\mathbb{R}$ is simply the collection of all $X \subseteq \mathbb{N}$ that are Dedekind cuts of $\mathbb{Q}$, i.e. nonempty, nonfull, downward closed along $<_{\mathbb{Q}}$, and without a greatest element.

Theorem (Weaver (2009a), Theorem 3.10)

$\mathbb{R}$ is a sequentially complete ordered field. Every sequentially complete ordered field is isomorphic to $\mathbb{R}$.

The classical proof works here: just show that in any sequentially complete ordered field $\mathbf{F}$, the countable subfield generated by 1 is isomorphic to $\mathbb{Q}$, and dense in $\mathbf{F}$.

Mathematics in CM

However, reasoning with genuinely uncountable collections will be difficult in CM. For example, we cannot conclude in general that the *image* of an uncountable function $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a set.

Mathematics in CM

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A more concrete example:

Definition

Let $\mathbf{X}$ be a metric space. A subset $\mathbf{U} \subseteq \mathbf{X}$ is *open* if there is a countable collection $P \subseteq \mathbf{X} \times \mathbb{R}^+$ such that $\mathbf{U} = \bigcup_{\langle x, r \rangle \in P} \text{ball}_r(x)$.

Theorem (Weaver (2009a), Theorem 4.33)

- *The union of a countable collection of open subsets of a metric space is open.*
- *The intersection of any finitely many open subsets of a separable metric space is open.*

Theorem (Heine–Borel/Bolzano–Weierstrass)

Let $\mathbf{K}$ be a separable subset of $\mathbb{R}$. Then the following are equivalent:

- ① $\mathbf{K}$ is closed and bounded;*
- ② $\mathbf{K}$ is compact;*
- ③ $\mathbf{K}$ is bounded and contains the limits of all of its Cauchy sequences;*
- ④ Every sequence in $\mathbf{K}$ has a subsequence which converges to a limit in $\mathbf{K}$.*

Theorem (Baire category theorem)

The intersection of any countable family of open dense sub-collections of a separable complete metric space is dense.

Theorem (Radon–Nikodym theorem)

Let μ be a separable finite measure on a set $\mathbf{X}$ and let ν be a (finite) signed measure on $\mathbf{X}$ that is absolutely continuous with respect to μ . Then there is a μ -integrable function $\mathbf{f} : \mathbf{X} \rightarrow \mathbb{R}$ such that

$$\nu(\mathbf{A}) = \int \mathbf{f} \cdot \chi_{\mathbf{A}} d\mu$$

for every μ -measurable set $\mathbf{A}$.

Theorem (Hahn–Banach theorem)

Let $\mathbf{E}$ be a separable Banach space, let $\mathbf{E}_0$ be a separable closed subspace, and let $\mathbf{f}_0 : \mathbf{E}_0 \rightarrow \mathbb{R}$ be a bounded linear functional on $\mathbf{E}_0$. Then $\mathbf{f}_0$ extends to a bounded linear functional $\mathbf{f}$ on $\mathbf{E}$ with $\|\mathbf{f}\| = \|\mathbf{f}_0\|$.

A choice principle?

Weaver's system lacks a genuine (uncountable) choice principle, which is crucial in a large portion of mainstream mathematics.

Weaver (2009a, section 2.3) proposes the following:

- Add a relation symbol $\prec$ between second-order objects that is a total ordering;
- The *transfinite induction* axiom

$$\forall X (\forall Y (Y \prec X \rightarrow \varphi(Y)) \rightarrow \varphi(X)) \rightarrow \forall X \varphi(X);$$

- the axiom of countable initial segments

$$\forall X \exists Z \forall Y (Y \prec X \rightarrow \exists n Y = (Z)_n).$$

Predicative transfinite induction

Question

What formulae should the transfinite induction scheme apply to?

Weaver (2009a) wrote:

[Transfinite induction] should only be asserted for formulas that do not contain $\prec$, for reasons having to do with the circularity involved in making sense of a relation that is well-ordered with respect to properties that are defined in terms of that relation.

But this restriction is merely superficial, because we can decidably comprehend

$$\mathbf{W} = \{\langle X, Y \rangle : X \prec Y\},$$

and use it instead of $\prec$ in the principle.

Zorn's lemma

We can prove in general:

Theorem (W. (2026a), Proposition 3.3.2)

Let $\varphi(X)$ be a decidable formula where X denotes a countable family, i.e. we have $\forall X (\varphi(X) \vee \neg\varphi(X))$. If φ is downward closed, i.e.

$$\forall X, Y ((\varphi(X) \wedge Y \subseteq X) \rightarrow \varphi(Y)),$$

and also closed under unions of countable chains, i.e.

$$\forall X (\forall Y, Z \in X (Y \subseteq Z \vee Z \subseteq Y) \wedge \forall Y \in X \varphi(Y)) \rightarrow \varphi(X),$$

then there exists some $\mathbf{X}$ satisfying $\forall X \subseteq \mathbf{X} \varphi(X)$, which is maximal among such subsets in the sense that

$$\forall Y (Y \not\subseteq \mathbf{X} \rightarrow \exists Z \subseteq \mathbf{X} \neg\varphi(Z \cup Y)).$$

Zorn's lemma

Proof of this Zorn's lemma only invokes transfinite induction on formulae with one second-order existential quantifier, no second-order universal quantifiers, and no third-order quantifiers.

By decidable comprehension, this is equivalent to the following single instance of transfinite induction:

$$\forall X (\forall Y \prec X \exists Z \langle Y, Z \rangle \in \mathbf{X} \rightarrow \exists Z \langle X, Z \rangle \in \mathbf{X}) \rightarrow \forall X \exists Z \langle X, Z \rangle \in \mathbf{X}.$$

This could be a “somewhat predicative” candidate in Weaver's standards.

Zorn's lemma

Corollary

$\mathbb{R}$ has a Hamel basis as a vector space over $\mathbb{Q}$;

Any inner product space, or any normed vector space over a complete $\mathbb{R}$ -normed field with a countable dense subset (e.g. any real or complex Banach space) also has a basis.

Corollary

Weaver's Hahn–Banach theorem generalises to extending a bounded linear functional $\mathbf{f}_0$ from separable close subspace $\mathbf{E}_0$ to nonseparable Banach space $\mathbf{E}$.

Consistency strength

When evaluating foundational formal systems in proof theory, we often compare their *consistency strength*, e.g.

- $\text{ZF} \vdash \text{Con}(\text{PA})$ (and thus $\text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{PA})$ but not the converse),
- yet $\text{Con}(\text{ZFC}) \leftrightarrow \text{Con}(\text{ZF})$.

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Definition (Ordinal analysis)

An *ordinal* $\alpha = \text{otyp}(\triangleleft)$ is a set-theoretic name assigned to characterise the length of a (transfinite) well-ordering. To a *nice enough* formal theory T , we shall assign a *proof-theoretic ordinal* $|T|$ as the least ordinal $\alpha = \text{otyp}(\triangleleft)$ with some canonically defined ordering $\triangleleft$ such that

$$T \equiv_{\Pi_2^0} \text{PA}^\dagger + \bigcup_a \text{TI}(\triangleleft \upharpoonright a)$$

where $\text{TI}(\triangleleft \upharpoonright a)$ denotes the transfinite induction scheme along initial segments of $\triangleleft$. (Rathjen, 1999)

Some ordinals and theories

$$\begin{aligned}\omega &:= \text{otyp}(1, 2, 3, \dots), \\ \omega + 1 &:= \text{otyp}(1, 2, 3, \dots, \omega), \\ \omega \cdot \omega &:= \omega + \omega + \omega + \dots.\end{aligned}$$

Then primitive recursive arithmetic, $|\text{PRA}| = \omega^\omega$;
and $|\text{HA}| = |\text{PA}| = \varepsilon_0 = \omega^{\omega^{\omega^{\dots}}}$.

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Veblen functions: Let $\varphi_0(\alpha) = \omega^\alpha$, and each φ_β enumerate the fixed points of $\{\varphi_\gamma\}_{\gamma < \beta}$, i.e.

$$\varphi_\beta(\alpha) = \min \{ \delta : \forall \gamma < \alpha \ \delta > \varphi_\beta(\gamma) \wedge \forall \gamma < \beta \ \delta = \varphi_\gamma(\delta) \}.$$

The systems of second-order arithmetic

- RCA_0 denotes the system surrounding the *recursive comprehension axiom*, with

$$|\text{RCA}_0| = \omega^\omega;$$

- WKL_0 denotes the axiom of *weak König lemma*, with

$$|\text{WKL}_0| = \omega^\omega;$$

- ACA_0 denotes the *arithmetic comprehension axiom*, with

$$|\text{ACA}_0| = \varepsilon_0 = \varphi_1(0);$$

- ATR_0 denotes the axiom of *arithmetic transfinite recursion*, with

$$|\text{ATR}_0| = \Gamma_0 = \sup_n \gamma_n$$

where $\gamma_0 = 0, \gamma_1 = \varphi_0(0), \dots, \gamma_{n+1} = \varphi_{\gamma_n}(0), \dots$;

- $\Pi_1^1\text{-CA}_0$ denotes the Π_1^1 -*comprehension axiom*, with a much higher proof-theoretic ordinal.

The systems of second-order arithmetic

$\Gamma_0 = |\text{ATR}_0|$ is usually treated as the “limit” of traditional Weylian predicativism, i.e. any predicatively justified system lies below this ordinal.

Question

What is the proof-theoretic ordinal of Weaver's system CM?

The systems of second-order arithmetic

$\Gamma_0 = |\text{ATR}_0|$ is usually treated as the “limit” of traditional Weylian predicativism, i.e. any predicatively justified system lies below this ordinal.

Question

What is the proof-theoretic ordinal of Weaver's system CM?

Consider the classical theory $\Sigma_1^1\text{-AC}$, consisting of

- ① arithmetic comprehension: for any arithmetic formula φ ,

$$\exists X \forall x (x \in X \leftrightarrow \varphi(x));$$

- ② Σ_1^1 -choice: for any Σ_1^1 formula φ ,

$$\forall n \exists X \varphi(n, X) \rightarrow \exists Z \forall n \varphi(n, (Z)_n);$$

- ③ full induction scheme on $\mathbb{N}$ for any second-order formula φ .

We know $|\Sigma_1^1\text{-AC}| = \varphi_{\varepsilon_0}(0)$. (Friedman, 1967)

A general notion of computation

A *partial combinatory algebra* (henceforth pca) is a domain $\mathcal{A}$ equipped partial binary operation $- \cdot - : \mathcal{A}^2 \rightarrow \mathcal{A}$. We write $ab \downarrow$ and say that ab *converges* when the value $a \cdot b$ is defined, and $ab \uparrow$ (ab *diverges*) otherwise.

Essential operations on a pca are $\mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1 \in \mathcal{A}$, such that

- ① $\mathbf{k}ab \downarrow = a$;
- ② $ac(bc) \downarrow \rightarrow \mathbf{s}abc \downarrow = ac(bc)$;
- ③ $\mathbf{p}_0(\mathbf{p}ab) \downarrow = a$ and $\mathbf{p}_1(\mathbf{p}ab) \downarrow = b$.

A pca over $\mathbb{N}$ additionally has

- ① $\mathbf{s}_{\mathbb{N}}n \downarrow = n + 1$ and $\mathbf{p}_{\mathbb{N}}(\mathbf{s}_{\mathbb{N}}n) \downarrow = n$;
- ② $\mathbf{d}_{\mathbb{N}}nmab \downarrow = \begin{cases} a & \text{if } n = m, \\ b & \text{if } n \neq m. \end{cases}$

A general notion of computation

Lemma (λ -abstraction)

Let $\mathcal{A}$ be a pca. For any (formal) term t over variables $x, \vec{y}$, there is another term, denoted as $\lambda x.t$ over variables $\vec{y}$ only, such that

① for any $\vec{b} \in \mathcal{A}^k$, $(\lambda x.t) [\vec{b}/\vec{y}] \downarrow$;

② for any $a \in \mathcal{A}$, $\vec{b} \in \mathcal{A}^k$,

$$t [a/x, \vec{b}/\vec{y}] \downarrow \rightarrow (\lambda x.t) [\vec{b}/\vec{y}] a \downarrow = t [a/x, \vec{b}/\vec{y}].$$

Lemma (Recursion theorem)

There is $\mathbf{r} \in \mathcal{A}$ such that for any $f, a \in \mathcal{A}$, $\mathbf{r}f \downarrow$ and

$$fa(\mathbf{r}f) \downarrow \rightarrow \mathbf{r}fa \downarrow = fa(\mathbf{r}f).$$

Proof. Build the Y-combinator as $t = \lambda fga.f a(gg)$ and $\mathbf{r} = \lambda f.tf(tf)$. □

Computing Σ_1^1 -definable functions

We work in Σ_1^1 -AC and fix a universal Π_1^0 -formula $\pi(e, m, a, b, X)$, so that correspondingly a universal Σ_1^1 -formula is given by $\exists X \pi(e, m, a, b, X)$, i.e.

Theorem (Simpson (2009), Lemma V.1.4, ACA_0)

For any Σ_1^1 -formula $\varphi(m, a, b)$, there is some $e \in \mathbb{N}$ such that

$$\forall m \forall a \forall b (\varphi(m, a, b) \leftrightarrow \exists X \pi(e, m, a, b, X)).$$

We can now define a pca on domain $\mathbb{N}$ by defining application as: $ea \downarrow = b$ if and only if b is the *unique* natural number satisfying

$$\exists X \pi(e_0, e_1, a, b, X),$$

where $e = \langle e_0, e_1 \rangle$.

Computing Σ_1^1 -definable functions

Lemma (Smn theorem, W. (2025), Proposition 2.3)

Given any Σ_1^1 -formula $\varphi(m, a, b)$ with all free variables indicated, then there exists $e \in \mathbb{N}$ such that for any $m, a \in \mathbb{N}$, $em \downarrow$ always holds, and $ema \downarrow = b$ if and only if $b \in \mathbb{N}$ is the unique natural number such that $\varphi(m, a, b)$ holds.

Corollary (Σ_1^1 -AC₀)

*Σ_1^1 -definable application indeed forms a pca on domain $\mathbb{N}$, i.e. we can explicitly find the required combinators **k** and **s**.*

The realisability model

We build a “model” of third-order arithmetic, by first choosing the domains of first-, second- and third-order variables.

- 1 The first-order variables range over $\mathbb{N}$;
- 2 The second-order variables range over

$$S_2 = \{d \in \mathbb{N} : \forall n \, dn \downarrow \in \{0, 1\}\},$$

which is a Δ_2^1 -subclass of $\mathbb{N}$;

- 3 The third-order variables range over

$$S_3 = \{d \in \mathbb{N} : \forall f \in S_2 \, df \downarrow \in \{0, 1\} \\ \wedge \forall e, f \in S_2 \, (e \equiv f \rightarrow de = df)\},$$

where $e \equiv f \Leftrightarrow \forall n \, en = fn$ is extensional equality on S_2 .

The realisability model

$d \Vdash t = s$	$\iff$	$t = s$ for any arithmetic terms t and s ,
$d \Vdash t \in_1 e$	$\iff$	$e \in S_2 \wedge et = 1$ for any arithmetic terms t ,
$d \Vdash e \in_2 f$	$\iff$	$e \in S_2 \wedge f \in S_3 \wedge fe = 1$,
$d \Vdash \varphi \wedge \psi$	$\iff$	$d = \langle d_0, d_1 \rangle \wedge d_0 \Vdash \varphi \wedge d_1 \Vdash \psi$,
$d \Vdash \varphi \vee \psi$	$\iff$	$d = \langle i, e \rangle \wedge ((i = 0 \wedge e \Vdash \varphi) \vee (i = 1 \wedge e \Vdash \psi))$,
$d \Vdash \neg \varphi$	$\iff$	$\forall e \neg e \Vdash \varphi$,
$d \Vdash \varphi \rightarrow \psi$	$\iff$	$\forall e (e \Vdash \varphi \rightarrow de \downarrow \Vdash \psi)$,
$d \Vdash \forall x \varphi(x)$	$\iff$	$\forall n dn \downarrow \Vdash \varphi(n)$,
$d \Vdash \exists x \varphi(x)$	$\iff$	$d = \langle d_0, d_1 \rangle \wedge d_1 \Vdash \varphi(d_0)$,
$d \Vdash \forall X \varphi(X)$	$\iff$	$\forall x \in S_2 dx \downarrow \Vdash \varphi(x)$,
$d \Vdash \exists X \varphi(X)$	$\iff$	$d = \langle d_0, d_1 \rangle \wedge d_0 \in S_2 \wedge d_1 \Vdash \varphi(d_0)$,
$d \Vdash \forall \mathbf{X} \varphi(\mathbf{X})$	$\iff$	$\forall x \in S_3 dx \downarrow \Vdash \varphi(x)$,
$d \Vdash \exists \mathbf{X} \varphi(\mathbf{X})$	$\iff$	$d = \langle d_0, d_1 \rangle \wedge d_0 \in S_3 \wedge d_1 \Vdash \varphi(d_0)$.

The realisability model

We say that “ φ is realised” if $\exists d \in \mathbb{N} \, d \Vdash \varphi$.

Lemma (Σ_1^1 -AC)

The induction and recursion schemes are realised:

$$\begin{aligned} & \varphi(0) \wedge \forall n (\varphi(n) \rightarrow \varphi(n+1)) \rightarrow \forall n \varphi(n), \\ & \forall n \forall X \exists Y \varphi(n, X, Y) \rightarrow \forall X \exists Z ((Z)_0 = X \wedge \forall n \varphi(n, (Z)_n, (Z)_{n+1})). \end{aligned}$$

Proof. Suppose that $a \Vdash \varphi(0)$ and $f \Vdash \forall n (\varphi(n) \rightarrow \varphi(n+1))$. We consider

$$g = \lambda n h. \begin{cases} a & \text{if } n = 0, \\ fm(hm) & \text{if } n = m + 1. \end{cases}$$

Then $\forall n \, \mathbf{r}gn \downarrow = gn(\mathbf{r}g) \Vdash \varphi(n)$ by induction.

Given $f \Vdash \forall n \forall X \exists Y \varphi(n, X, Y)$, we can likewise compute the family $\{(Z)_n\}_n$ by recursion. □

The realisability model

Lemma (Σ_1^1 -AC₀)

The decidable comprehension axioms are realised:

$$\begin{aligned}\forall n (\varphi(n) \vee \neg \varphi(n)) &\rightarrow \exists X \forall n (n \in X \leftrightarrow \varphi(n)), \\ \forall X (\varphi(X) \vee \neg \varphi(X)) &\rightarrow \exists \mathbf{X} \forall X (X \in \mathbf{X} \leftrightarrow \varphi(X));\end{aligned}$$

Proof. If $f \Vdash \forall n (\varphi(n) \vee \neg \varphi(n))$, then

$$g = \lambda n.1 - \mathbf{p}_0(fn)$$

is simply the object in S_2 representing the set X . Same applies to the third-order case. □

The realisability model

Lemma (Σ_1^1 -AC₀)

The numerical omniscience principle is realised:

$$\forall n (\varphi(n) \vee \psi(n)) \rightarrow (\forall n \varphi(n) \vee \exists n \psi(n)).$$

Proof. We take advantage of the fact that our functions just need to be Σ_1^1 -definable. Given $f \Vdash \forall n (\varphi(n) \vee \psi(n))$, we can define $gn \downarrow = i$ if and only if

$$(i = 0 \wedge \forall n \mathbf{p}_0(fn) = 0) \vee (i = 1 \wedge \exists n \mathbf{p}_0(fn) = 1)$$

to decide the correct disjunct in the consequent. □

The realisability model

Lemma (Σ_1^1 -AC₀)

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It follows that for any axiom φ of CM, Σ_1^1 -AC $\vdash \exists d d \Vdash \varphi$. It follows that

$$(\text{CM} \vdash \varphi) \rightarrow (\Sigma_1^1\text{-AC} \vdash \exists d d \Vdash \varphi),$$

and substituting $\varphi = \perp$ gives $\text{Con}(\Sigma_1^1\text{-AC}) \rightarrow \text{Con}(\text{CM})$.

Ordinal analysis of CM

We have interpreted CM inside $\Sigma_1^1\text{-AC}$.

Let $\Sigma_1^1\text{-AC}^i$ denote the intuitionistic part of $\Sigma_1^1\text{-AC}$, i.e. consequences of the same axioms but without excluded middle. Trivially $\text{CM} \vdash \Sigma_1^1\text{-AC}^i$.

Aczel (1977) checked that $|\Sigma_1^1\text{-AC}| = |\Sigma_1^1\text{-AC}^i|$.

Thus we also have

$$\text{Con}(\Sigma_1^1\text{-AC}) \leftrightarrow \text{Con}(\text{CM}),$$

and $|\text{CM}| = |\Sigma_1^1\text{-AC}| = \varphi_{\varepsilon_0}(0)$.

Interpreting a global well-ordering

We need to sort the class S_2 into a well-ordering. Observe that for $f \in S_2$,

$$fn \downarrow = 0 \leftrightarrow \neg fn \downarrow = 1$$

is Δ_1^1 -definable.

Given any countable well-ordering α , we define $\text{TJ}^0(X) = X$ and

$$\text{TJ}^\alpha(X) = \text{TJ}\left(\left\{\text{TJ}^\beta(X)\right\}_{\beta < \alpha}\right)$$

for $\alpha > 0$. Then the Kleene–Souslin theorem states:

Theorem (Simpson (2009), Theorem VIII.3.19, ATR_0)

For any $Z \subseteq \mathbb{N}$, Z is Δ_1^1 -definable in parameter X if and only if there exists a recursive well-ordering α such that Z is Turing reducible to $\text{TJ}^\alpha(X)$.

Interpreting a global well-ordering

We can effectivise the Kleene–Souslin theorem:

Theorem (W. (2025), Theorem 4.3, ATR_0)

There exists a function $\mathbf{h}$ in the Σ_1^1 -pca such that, given any Δ_1^1 -set Z that is defined by its characteristic function $f \in S_2$, we have $\mathbf{h}f \downarrow = \langle \alpha, e \rangle$, such that the oracle machine $\Phi_e^{\text{TJ}^\alpha(\emptyset)}$ computes the set Z .

Additionally, for $f \equiv g \in S_2$, we have $\mathbf{h}f = \mathbf{h}g$.

We can well-order S_2 by defining $f \prec g$ if and only if $\mathbf{h}f < \mathbf{h}g$ lexicographically.

Thus, we can add the realisability condition $d \Vdash f \prec g$ if and only if $f \prec g$ is true in the sense above.

Interpreting transfinite induction

Let *Bar induction*, BI, denote the second-order axiom scheme

$$\text{WO}(\triangleleft) \wedge \forall n (\forall m \triangleleft n \varphi(m) \rightarrow \varphi(n)) \rightarrow \forall n \varphi(n)$$

for arbitrary formulae φ . We have

$$\text{BI} \Rightarrow \text{ATR} \Rightarrow \Sigma_1^1\text{-AC},$$

and $|\text{BI}| = |\text{ID}_1| = \text{Bachmann–Howard ordinal}$ (Buchholz and Pohlers, 1978).

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Lemma (BI)

The transfinite induction scheme for the global well-ordering is realised for arbitrary third-order formulae φ :

$$\forall X (\forall Y (Y \prec X \rightarrow \varphi(Y)) \rightarrow \varphi(X)) \rightarrow \forall X \varphi(X).$$

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Proof sketch. Suppose that $f \Vdash \forall X (\forall Y (Y \prec X \rightarrow \varphi(Y)) \rightarrow \varphi(X))$, then

$$\forall x \in S_2 \text{ r}(\lambda ah. fa(\lambda bc. hb))x \downarrow \Vdash \varphi(x),$$

though verifying this needs some technical arguments to work around functional extensionality. □

Interpreting countable initial segments

Lemma (ATR_0)

The axiom of countable initial segments is realised:

$$\forall X \exists Z \forall Y (Y \prec X \rightarrow \exists n Y = (Z)_n).$$

Proof. Suppose that $x \in S_2$ is computed by the oracle machine $\Phi_e^{\text{TJ}^\alpha(\emptyset)}$. We can simply take the initial segment

$$\left\{ \Phi_{e'}^{\text{TJ}^\alpha(\emptyset)} : e' \in \mathbb{N}, \Phi_{e'}^{\text{TJ}^\alpha(\emptyset)} \text{ terminates on all inputs} \right\}$$

which is countable (because there are only countably many Turing machines) and also Σ_1^1 -definable. □

Ordinal analysis of CM + full GWO

We have checked that for any axiom φ of CM + full GWO,
 $\text{BI} \vdash \exists d \, d \Vdash \varphi$. Thus

$$\text{Con}(\text{BI}) \rightarrow \text{Con}(\text{CM}).$$

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For the converse, we check that the system ID_1^i of intuitionistic first-order inductive definitions is interpreted in CM + full GWO.

We say $\varphi(X, n)$ is a *strictly positive first-order formula*, if it only contains:

- ① any atomic formulae in the language of PA as well as $t \in X$ for any first-order term t ;
- ② $\perp$ and connectives $\wedge, \vee, \rightarrow$, with the restriction that X cannot occur in an antecedent;
- ③ first-order quantifiers $\forall n$ and $\exists n$.

First-order inductive definitions

Theorem (Inductive definitions)

For any strictly positive first-order formula $\varphi(X, n)$, there exists a formula $\theta_\varphi(n)$ such that

$$\forall n (\varphi(\theta_\varphi, n) \rightarrow \theta_\varphi(n))$$

holds, and also for any formula $\eta(n)$,

$$\forall n (\varphi(\eta, n) \rightarrow \eta(n)) \rightarrow \forall n (\theta_\varphi(n) \rightarrow \eta(n)).$$

Proof sketch. We define a family of sets $\{A_X\}_X$ indexed by second-order objects, such that

$$A_X = \left\{ n \in \mathbb{N} : \varphi\left(\bigcup_{Y \prec X} A_Y, n\right) \right\}.$$

It then suffices to take $\theta_\varphi(n) = \exists X n \in A_X$. The verifications follow any textbook proof for inductive definitions in systems like BI or KP. $\square$

Ordinal analysis of CM + full GWO

We have interpreted CM + full GWO inside BI and also interpreted ID_1^i inside CM + full GWO.

Since $|ID_1^i| = |ID_1| = |BI| =$ Bachmann–Howard ordinal (Buchholz and Pohlers, 1978), we have

$$|CM + \text{full GWO}| = \text{Bachmann–Howard ordinal}.$$

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$$|\text{CM} + \text{full GWO}| = \text{Bachmann-Howard ordinal}.$$

Let $\Sigma\text{-GWO}$ denote the “predicative” fragment with the single transfinite induction axiom

$$\forall X (\forall Y \prec X \exists Z \langle Y, Z \rangle \in \mathbf{X} \rightarrow \exists Z \langle X, Z \rangle \in \mathbf{X}) \rightarrow \forall X \exists Z \langle X, Z \rangle \in \mathbf{X}.$$

What is its strength then?

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What is its strength then? In order to realise it, we essentially need to prove

$$\forall x \in S_2 \text{ } \mathbf{rg}x \downarrow \Vdash \exists Z \langle X, Z \rangle \in \mathbf{X}$$

by transfinite induction — this requires at least Π_2^1 -transfinite induction.

Transfinite computation, another way

Let KP^- denote the Kripke–Platek set theory without foundation, consisting of the axioms of extensionality, pairing, union, Δ_0 -separation, Δ_0 -collection and infinity. We add Σ_1 -set induction, Σ_1 -SI,

$$\forall x (\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x)$$

for any Σ_1 -formula φ .

This allows us to define Gödel's constructible universe L as

$$\begin{aligned} L_0 &= \emptyset, \\ L_{\alpha+1} &= \text{Def}(L_\alpha), \\ L_\lambda &= \bigcup_{\alpha < \lambda} L_\alpha \quad \text{for limit } \lambda, \\ L &= \bigcup_{\alpha \in \text{Ord}} L_\alpha. \end{aligned}$$

Transfinite computation, another way

There is a Σ_1 -definable well-ordering on L , which we can represent by a class bijection $H : \text{Ord} \rightarrow L$ (Devlin, 1984, section II.2). If we add the axiom $V = L$, this becomes a well-ordering of the entire universe.

Let $\text{Tr}(\ulcorner \varphi \urcorner, m, a, b, x)$ be any Δ_1 -definable truth predicate for a fixed coding of arbitrary Δ_0 -formula φ with *precisely* four free variables, so that

$$\exists x \text{Tr}(t, m, a, b, x)$$

is essentially a universal Σ_1 -formula. We can now turn the entire universe into a pca by defining $ea \downarrow = b$ if and only if the following Σ_1 -evaluation formula $\varepsilon(e, a, b)$ is satisfied:

$$e = \langle t, m \rangle \wedge \exists x \left(\text{Tr}(t, m, a, b, x) \wedge \forall \beta \in H^{-1}(\langle b, x \rangle) \left(H(\beta) = \langle b', x' \rangle \rightarrow \neg \text{Tr}(t, m, a, b', x') \right) \right),$$

i.e. b, x is the H -least pair satisfying $\text{Tr}(t, m, a, b, x)$.

Transfinite computation, another way

Lemma (Smn theorem, W. (2026b), Lemma 2.4)

Given any Σ_1 -formula $\phi(m, a, b)$, there exists e such that for any set m, a , $em\downarrow$ always hods, and $ema\downarrow = b$ if and only if there exists some set b satisfying $\phi(m, a, b)$, in which case also

$$\varphi(m, a, ema).$$

Theorem ($KP^- + \Sigma_1\text{-SI} + V = L$)

Σ_1 -definable application forms a pca on domain V .

The realisability model

We model CM now in set theory, by letting

- ① The first-order variables range over ω ;
- ② The second-order variables range over the Δ_0 -class $\mathcal{P}(\omega)$;
- ③ The third-order variables range over

$$S_3 = \{d : \forall x \subseteq \mathbb{N} \, dx \downarrow \in \{0, 1\}\}.$$

The realisability conditions are essentially the same, except we interpret the global well-ordering on second-order objects as

$$d \Vdash x \prec y \iff H^{-1}(x) \in H^{-1}(y).$$

The realisability model

The interpretations of the axioms of CM are also essentially the same, except we need the set-forming lemma:

Lemma

Suppose f satisfies $\forall n \in \omega \, fn \downarrow \in \{0, 1\}$, then there exists a set $X \subseteq \omega$ such that

$$x = \{n \in \omega : fn \downarrow = 1\}.$$

Proof. We know that

$$fn \downarrow = 1 \leftrightarrow \neg fn \downarrow = 0$$

is Δ_1 . KP^- proves Σ_1 -collection, which implies Δ_1 -separation. Thus x is a set. □

Interpreting countable initial segments

Lemma ($KP^- + \Sigma_1\text{-SI} + V = L$)

The axiom of countable initial segments is realised:

$$\forall X \exists Z \forall Y (Y \prec X \rightarrow \exists n Y = (Z)_n).$$

Proof. Fix any $x \subseteq \omega$. By $V = L$, $x \in L$, thus by the Gödel condensation lemma, there exists a countable $\alpha \in \text{Ord}$ such $x \in L_\alpha$. Let $\alpha_0 \in \text{Ord}$ be the least ordinal such that $x \in L_{\alpha_0}$, then L_{α_0} is countable and we can take the H -least bijection

$$f : \omega \rightarrow L_{\alpha_0} \cap \mathcal{P}(\omega).$$

It suffices to take the initial segment $\{z_n\}_{n \in \omega}$ where each $z_n = f(n)$. □

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Lemma ($KP^- + \Sigma_1\text{-SI} + V = L$)

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Thus $KP + V = L + \text{full } \omega\text{-IND}$ realises $\text{CM} + \text{full GWO}$.

Interpreting Σ -transfinite induction

Lemma (KP + Σ_1 -SI + $V = L$)

The transfinite induction axiom for Σ -GWO is realised:

$$\forall X (\forall Y \prec X \exists Z \langle Y, Z \rangle \in \mathbf{X} \rightarrow \exists Z \langle X, Z \rangle \in \mathbf{X}) \rightarrow \forall X \exists Z \langle X, Z \rangle \in \mathbf{X}.$$

Proof sketch. Fix $h \in S_3$ representing $\mathbf{X}$, and apply the same construction as in the BI proof. Notice that we just need to prove

$$\forall x \subseteq \omega (\mathbf{p}_0(\mathbf{rg}x) \downarrow \subseteq \omega \wedge h \langle x, \mathbf{p}_0(\mathbf{rg}x) \rangle = 1)$$

by transfinite induction (on the ordinal $H^{-1}(x)$), where our evaluation formula for the pca is Σ_1 . Thus this follows from Σ_1 -set induction. $\square$

Ranked tree interpretation

For the converse, i.e. to interpret the fragment of KP into $\text{CM} + \Sigma\text{-GWO}$, we need a variant of the tree interpretation from Simpson (2009, section VII.3). We say that some $T \subseteq [\mathbb{N}]^{<\mathbb{N}}$ is a *ranked tree* if

- ① T is a tree, i.e. T is nonempty and for any finite sequences $s \sqsubseteq t$, if $t \in T$ then $s \in T$;
- ② T has a *ranking object*, i.e. some $R_T \subseteq \mathbb{N}$ such that for any $s \sqsubseteq t \in T$, we have

$$(R_T)_s \succ (R_T)_t.$$

Essentially, since $\prec$ is a well-ordering, the existence of a ranking object ensures that T is well-founded, i.e. T has no infinite path.

Lemma ($\text{CM} + \Sigma\text{-GWO}$)

Every ranked tree has a unique bisimulation.

Ranked tree interpretation

Given ranked tree T, S , the following trees are also ranked:

- 1 The pair tree

$$\{\langle \rangle\} \cup \{\langle 0 \rangle \frown t : t \in T\} \cup \{\langle 1 \rangle \frown s : s \in S\};$$

- 2 The union tree

$$\{\langle \rangle\} \cup \{\langle \langle i, j \rangle \rangle \frown t : \langle i, j \rangle \frown t \in T\};$$

- 3 The ω -tree

$$\left\{ t \in \mathbb{N}^{<\mathbb{N}} : t \text{ is strictly decreasing} \right\};$$

- 4 The comprehension tree on a Δ_1^1 -condition $\varphi(T, i)$

$$\{\langle \rangle\} \cup \{\langle i \rangle \frown t : \langle i \rangle \frown t \in T, \varphi(T, i)\}.$$

The recursion scheme (specifically Σ_1^1 -choice) also interprets Δ_0 -collection.

Ordinal analysis of $\text{CM} + \Sigma\text{-GWO}$

We have shown that $\text{KP}^- + \Sigma_1\text{-SI} + V = L + \text{full } \omega\text{-IND}$ interprets $\text{CM} + \Sigma\text{-GWO}$, and also $\text{CM} + \Sigma\text{-GWO}$ interprets $\text{IKP}^- + \Sigma_1\text{-SI} + \text{full } \omega\text{-IND}$.

$V = L$ does not affect the proof-theoretic strength here, and Avigad (2000) checked that the classical and intuitionistic variants of fragments of KP have equivalent proof-theoretic strength.

It is known (Rathjen, 1993) that

$$|\text{KP}^- + \Sigma_1\text{-SI} + \text{full } \omega\text{-IND}| = \varphi_{\varphi_{\varepsilon_0}(0)}(0),$$

thus we also have

$$|\text{CM} + \Sigma\text{-GWO}| = \varphi_{\varphi_{\varepsilon_0}(0)}(0).$$

The Feferman–Schütte limit

Start with a family of infinitary proof systems RA_α of *ramified analysis*, where the ordinal α essentially dictates that “any proof of transfinite height $< \alpha$ is allowed”.

Definition

The *autonomous ordinals*, AUT is the smallest set of recursive ordinals satisfying the following:

- 1 0 is in AUT ;
- 2 If $\beta \in \text{AUT}$ and $\gamma < \beta$ then $\gamma \in \text{AUT}$;
- 3 if $\alpha \in \text{AUT}$ and $\text{RA}_\alpha \vdash \text{WO}(\delta)$ then $\delta \in \text{AUT}$.

Theorem (Feferman (1964); Schütte (1965))

AUT contains precisely those ordinals $\alpha < \Gamma_0$.

A reasonable criticism

Howard (1996, p. 280) raises the following question, which Weaver supports through his conceptualism:

Recall we have ordinals $\gamma_0 = 0, \gamma_1 = \varphi_0(0), \dots, \gamma_{n+1} = \varphi_{\gamma_n}(0), \dots$ and $\Gamma_0 = \sup_n \gamma_n$. The Feferman–Schütte argument essentially reveals to us that “if γ_n is well-ordered, then a proof system of the height γ_n provides a predicative way of justifying that γ_{n+1} is well-ordered”. Why then, is the predicativist unable to *reflect* on all these justifications and conclude that $\Gamma_0 = \bigcup_n \gamma_n$ (and thus $\Gamma_0 + 1$, etc.) is well-ordered?

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If we admit the conceptualist idea that any countable transfinite process (of reasoning) should be acceptable, then it seems that we should add an extra rule to the autonomous progression:

- ④ if $(\alpha_n)_{n \in \mathbb{N}}$ is a computable (or primitive recursive) enumeration of autonomous ordinals, then $\sup_{n \in \mathbb{N}} \alpha_n \in \text{AUT}$.

Then the limit of the autonomous system is expected to be much higher.

Weaver's strong systems

Given a theory $\mathcal{S}$ in language L , for each α , let L_α be the language L augmented by a family of truth predicates $\{T_\beta\}_{\beta < \alpha}$, and let $\mathcal{S}_\alpha$ be the theory in $L_{\alpha+1}$ consisting of

- The axioms of $\mathcal{S}$;
- The axiom " $T_\alpha(\ulcorner A \urcorner)$ for any axiom A of $\mathcal{S}$ ";
- The axiom " $T_\alpha(\ulcorner A \urcorner) \wedge T_\alpha(\ulcorner B \urcorner) \rightarrow T_\alpha(\ulcorner C \urcorner)$ for any formulae A, B, C in L_α , for which there is a deduction rule that infers C from A and B ";
- The axiom " $T_\alpha(\ulcorner A(0) \wedge \forall n (A(n) \rightarrow A(n+1)) \urcorner) \rightarrow \forall n A(n) \urcorner$ for any formula $A(n)$ in L_α ";
- The axiom " $\forall n T_\alpha(\ulcorner A(\hat{n}) \urcorner) \leftrightarrow T_\alpha(\ulcorner \forall n A(n) \urcorner)$ for any formula $A(n)$ in L_α ";
- The axiom scheme " $\bar{A} \leftrightarrow T_\alpha(\ulcorner A \urcorner)$ " for every formula A in L_α , where $\bar{A}$ is the universal closure of A ;
- The axiom " $T_\alpha(\ulcorner \bar{A} \urcorner \leftrightarrow T_\beta(\ulcorner A \urcorner) \urcorner)$ for any $\beta < \alpha$ and any formula A in L_β ";
- For every $\beta < \alpha$, an axiom " $T_\beta(\ulcorner A \urcorner) \leftrightarrow T_\alpha(\ulcorner A \urcorner)$ for any formula A in L_β ".

Weaver's strong systems

Let $\text{Tarski}(\mathcal{S})$ denote following axioms:

- The axioms of $\mathcal{S}$;
- The axiom $\text{Prog}(\text{Acc}) := \forall \beta < \alpha \text{ Acc}(\beta) \leftrightarrow \text{Acc}(\alpha)$;
- The unrestricted induction scheme for all formulae in the language of $\mathcal{S}$ augmented by the symbol Acc ;
- For each α and every axiom A of $\mathcal{S}_\alpha$ the statement “ $\text{Acc}(\hat{\alpha}) \rightarrow A$ ”.

Lemma (Weaver (2022), Corollary 5.5)

For any $n \geq 2$, $\text{Tarski}(\mathcal{S})$ plus transfinite induction up to γ_n for the language of $\text{Tarski}(\mathcal{S})$ proves transfinite induction up to γ_{n+1} for the language of $\mathcal{S}$.

Corollary

$\text{Tarski}^\omega(\text{PA})$ proves transfinite induction up to Γ_0 for all formulae in its language.

Weaver's strong systems

Weaver (2009d) claimed even stronger systems, namely

- $\text{Tarski}_{\Gamma_0}^\omega(\text{PA})$ proves transfinite induction up to any ordinal less than Γ_0 for all formulae in its language;
- $\text{Tarski}_{\kappa^\kappa}^\omega(\text{PA})$ proves transfinite induction up to any ordinal less than $\theta_{\Omega^2}(0)$ for all formulae in its language;
- $\text{Tarski}_{\lambda^{\lambda^\omega}}^\omega(\text{PA})$ proves transfinite induction up to any ordinal less than $\theta_{\Omega^\omega}(0)$ for all formulae in its language;

and declare that the process can continue to at least reach the large Veblen ordinal $\theta_{\Omega^\Omega}(0)$.

But these are highly complicated, non-uniform arguments that have not yet undergone peer review; and it is unclear how the strength from these truth predicates truly affect the treatment of mainstream mathematics within.

Dziękuję!

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