Indexed hierarchies and internal structure in the ordinal class in intuitionistic set theories

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Ordinals and ordinal-indexed hierarchies

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$$\mathrm{Ord} = \{\alpha : \alpha \text{ is transitive} \land \forall \beta \in \alpha \ \beta \text{ is transitive}\}.$$

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Fix a set-like partial order $\mathscr{R}$ on the class Tr of all transitive sets, which is no coarser than $\subseteq$ yet no finer than $\in$, i.e.

$$x = y \vee x \in y \Longrightarrow x \mathscr{R} y; \quad x \mathscr{R} y \Longrightarrow x \subseteq y,$$

and we consider the $\mathscr{R}$*-predecessor* set

$$P_{\mathscr{R}}(x) = \{y : y \mathscr{R} x\}.$$

E.g. when $\mathscr{R} = \subseteq$, $P_{\mathscr{R}}(x) = \mathcal{P}(x)$; when $\mathscr{R} = \in$, $P_{\mathscr{R}}(x) = x^+ = x \cup \{x\}$. We can also get the first-order definable subsets $\mathrm{Def}(x)$, etc.

Ordinals and ordinal-indexed hierarchies

Fix a set-like partial order $\mathscr{R}$ on the class Tr of all transitive sets, which is no coarser than $\subseteq$ yet no finer than $\in$. We can define

$$\begin{aligned} H_{\mathscr{R}}(\varnothing) &= \varnothing, \\ H_{\mathscr{R}}(\alpha^{+}) &= P_{\mathscr{R}}(H_{\mathscr{R}}(\alpha)), \\ H_{\mathscr{R}}(\lambda) &= \bigcup_{\alpha \in \lambda} H_{\mathscr{R}}(\alpha) \quad \text{for limit } \lambda. \end{aligned}$$

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Alternatively, for any set x, we can define

$$H_{\mathscr{R}}(x) = \bigcup_{y\in x} P_{\mathscr{R}}(H_{\mathscr{R}}(y)).$$

One easily verifies that this gives the same results on ordinals.

Ordinals and ordinal-indexed hierarchies

Fix a set-like partial order $\mathscr{R}$ on the class Tr of all transitive sets, which is no coarser than $\subseteq$ yet no finer than $\in$.

Question

Is there a maximal class $O_{\mathscr{R}} \subseteq V$ on which $H_{\mathscr{R}}$ is an injection?

'Maximal' means for any $x \in V$, there exists $y \in O_{\mathscr{R}}$ such that $H_{\mathscr{R}}(x) = H_{\mathscr{R}}(y)$.

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Answer (ZF). Always take $O_{\mathscr{R}} = \mathrm{Ord}$.

Injectivity follows from linearity of ordinals (note $x \in y$ implies $H_{\mathscr{R}}(x) \in H_{\mathscr{R}}(x^+) \subseteq H_{\mathscr{R}}(y)$).

For any $x \in V$, simply observe $H_{\mathscr{R}}(x) = H_{\mathscr{R}}(\mathrm{rank}(x))$.

Intuitionistic set theory

Intuitionistic Zermelo–Fraenke set theory IZF is:

ZF, minus Excluded Middle, with Axiom of Foundation/Regularity replaced by Set Induction:

$$\forall x\,(\forall y \in x\ \varphi(y) \to \varphi(x)) \to \forall x\ \varphi(x).$$

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Wellknown folklore

IZF *is consistent with* Ord *being non-linear, or formally*

$$\exists \alpha, \beta \in \mathrm{Ord}\,(\beta \subseteq \alpha \wedge \beta \notin \alpha \wedge \beta \neq \alpha)\,.$$

This is easily implementable in multiple types of 'intuitionistic models', e.g. models on Kripke frames, realisability models or Heyting-valued models.

Intuitionistic set theory

Corollary

IZF *is consistent with the class function* $\alpha \mapsto V_\alpha$ *being non-injective.*

Proof. Take $\alpha, \beta \in \mathrm{Ord}$ such that $\beta \subseteq \alpha$, $\beta \notin \alpha$ and $\beta \neq \alpha$. Consider

$$V_{\alpha^+} = \mathcal{P}(V_\alpha).$$

But $V_\beta \subseteq V_\alpha$, i.e. $\mathcal{P}(V_\beta) \subseteq \mathcal{P}(V_\alpha)$, thus

$$V_{\alpha^+ \cup \beta^+} = \mathcal{P}(V_\alpha) \cup \mathcal{P}(V_\beta) = V_{\alpha^+}.$$

Clearly $\beta \notin \alpha^+$, so $\alpha^+ \neq \alpha^+ \cup \beta^+$. □

The intuition

Question

Is there a maximal class $O_{\mathscr{R}} \subseteq V$ on which $H_{\mathscr{R}}$ is an injection?

It is still reasonable to expect $O_{\mathscr{R}} \subseteq \mathrm{Ord}$. Ordinals in $O_{\mathscr{R}}$ should satisfy some additional 'closure properties'.

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It is still reasonable to expect $O_{\mathscr{R}} \subseteq \mathrm{Ord}$. Ordinals in $O_{\mathscr{R}}$ should satisfy some additional 'closure properties'.

Example (Paul Taylor, 1996): We say an ordinal $\alpha \in \mathrm{Ord}$ is *plump* if

- every $\beta \in \alpha$ is plump;
- for every $\gamma \in \mathrm{Ord}$, if γ is plump and $\gamma \subseteq \beta$ for some $\beta \in \alpha$, then $\gamma \in \alpha$.

As it turns out, the class of plump ordinals is precisely a maximal class on which $\alpha \mapsto V_\alpha$ is injective.

The intuition

A similar idea comes from Robert Lubarsky, *Intuitionistic L* (1993):

Theorem

Although it is open whether $\mathrm{IZF} \vdash \mathrm{Ord} \subseteq L$*, we can avoid it and still prove* $\mathrm{IZF} \vdash (V = L)^L$*.*

Proof. For any $\alpha \in \mathrm{Ord}$, we construct $\alpha^* \in L$ as

$$\alpha^* = \left\{ \beta \in \mathrm{Ord} : L_{(\alpha_{\mathrm{aug}} +_{\mathrm{H}} n)^-} \vDash \mathrm{def}(L_\beta) \subseteq L_\alpha \right\},$$

such that $L_\alpha = L_{\alpha^*} \in L$. □

Note that $\mathrm{def}(L_\beta) \subseteq L_\alpha$ can be equivalently written as $L_\beta \in L_\alpha$, since if $L_\beta \in \mathrm{def}(L_\gamma)$ for some $\gamma \in \alpha$, then also

$$\mathrm{def}(L_\beta) \subseteq \mathrm{def}(L_\gamma) \subseteq L_\alpha.$$

The formal construction

We now directly define

$$O_{\mathscr{R}} = \{\alpha \in \mathrm{Ord} : \forall \beta \in \alpha^{+}\ \forall \gamma \in \mathrm{Ord} \\ \big(H_{\mathscr{R}}(\gamma) \in H_{\mathscr{R}}(\beta) \to \exists \gamma' \in \beta\ H_{\mathscr{R}}\big(\gamma'\big) = H_{\mathscr{R}}(\gamma)\big)\}.$$

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Proposition

$H_{\mathscr{R}}$ is injective on $O_{\mathscr{R}}$.

Proof. Use induction. Suppose $\beta, \beta' \in O_{\mathscr{R}}$ and $H_{\mathscr{R}}(\beta) = H_{\mathscr{R}}(\beta')$, we want to show $\beta = \beta'$. For $\gamma \in \beta$,

$$H_{\mathscr{R}}(\gamma) \in H_{\mathscr{R}}(\beta) = H_{\mathscr{R}}(\beta'),$$

i.e. $H_{\mathscr{R}}(\gamma) = H_{\mathscr{R}}(\gamma')$ for $\gamma' \in \beta'$. By inductive hypothesis, $\gamma = \gamma' \in \beta'$. The converse is symmetrical. □

$\mathscr{R}$-successors

Are there many elements in $O_{\mathscr{R}}$?

We want to first find the $\mathscr{R}$-successor of $\alpha \in O_{\mathscr{R}}$, i.e. some $\alpha^{\mathscr{R}+} \in O_{\mathscr{R}}$ such that

$$H_{\mathscr{R}}\left(\alpha^{\mathscr{R}+}\right) = H_{\mathscr{R}}(\alpha^{+}) = P_{\mathscr{R}}(H_{\mathscr{R}}(\alpha)).$$

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We consider the set

$$s = \{\{\gamma \in \alpha : H_{\mathscr{R}}(\gamma) \in x\} : x \in P_{\mathscr{R}}(H_{\mathscr{R}}(\alpha))\}.$$

For any $\beta \in \mathrm{Ord}$, if $H_{\mathscr{R}}(\beta) \in P_{\mathscr{R}}(H_{\mathscr{R}}(\alpha))$, then for each $\gamma \in \beta$, there is $\gamma' \in \alpha$ such that $H_{\mathscr{R}}(\gamma') = H_{\mathscr{R}}(\gamma)$. Thus

$$H_{\mathscr{R}}(\beta) = \bigcup_{\gamma \in \beta} P_{\mathscr{R}}(H_{\mathscr{R}}(\gamma)) = \bigcup_{\gamma' \in t} P_{\mathscr{R}}\left(H_{\mathscr{R}}\left(\gamma'\right)\right) = H_{\mathscr{R}}(t)$$

for $t = \{\gamma \in \alpha : H_{\mathscr{R}}(\gamma) \in H_{\mathscr{R}}(\beta)\} \in s$. Easy to verify $t \in O_{\mathscr{R}}$.

$\mathscr{R}$-successors

For $\alpha \in O_{\mathscr{R}}$, consider the set

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it suffices to define $\alpha^{\mathscr{R}+} = \{\beta \in s : \beta \in O_{\mathscr{R}}, H_{\mathscr{R}}(\beta)\,\mathscr{R}\,H_{\mathscr{R}}(\alpha)\}$.

Lemma

For any $\alpha \in O_{\mathscr{R}}$, $\alpha^{\mathscr{R}+} \in O_{\mathscr{R}}$ is the least element in $O_{\mathscr{R}}$ that contains α.

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Lemma

For $s \subseteq O_{\mathscr{R}}$, $\sup s = \bigcup s \in O_{\mathscr{R}}$.

So we can define $\mathrm{rk}_{\mathscr{R}} : V \to O_{\mathscr{R}}$ as

$$\mathrm{rk}_{\mathscr{R}}(x) = \bigcup_{y \in x} \mathrm{rk}_{\mathscr{R}}(y)^{\mathscr{R}+}.$$

Maximality of $O_{\mathscr{R}}$

$$\mathrm{rk}_{\mathscr{R}}(x) = \bigcup_{y \in x} \mathrm{rk}_{\mathscr{R}}(y)^{\mathscr{R}+}.$$

Using $H_{\mathscr{R}}(\alpha^{\mathscr{R}+}) = P_{\mathscr{R}}(H_{\mathscr{R}}(\alpha))$, we prove by induction that:

Proposition

For any set x,

$$H_{\mathscr{R}}(x) = H_{\mathscr{R}}(\mathrm{rk}_{\mathscr{R}}(x)).$$

Corollary

For any set x, there exists $\alpha \in O_{\mathscr{R}}$ such that $H_{\mathscr{R}}(x) = H_{\mathscr{R}}(\alpha)$, i.e. $O_{\mathscr{R}}$ is a maximal class on which $H_{\mathscr{R}}$ is injective.

Consistency of class separations

- ZF proves that $O_{\mathscr{R}} = \mathrm{Ord}$ for all $\mathscr{R}$.
- When $\mathscr{R} = \in$, $H_{\mathscr{R}}(x) = \mathrm{rk}_{\mathrm{Ord}}(x)$, so we always have $O_{\in} = \mathrm{Ord}$.

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- We showed at the beginning that IZF is consistent with $\alpha \neq \beta \wedge V_\alpha = V_\beta$, so it is consistent with $O_{\subseteq} \neq \mathrm{Ord} = O_{\in}$.

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- We showed at the beginning that IZF is consistent with $\alpha \neq \beta \wedge V_\alpha = V_\beta$, so it is consistent with $O_{\subseteq} \neq \mathrm{Ord} = O_{\in}$.
- Let $\mathscr{R}_{\mathrm{def}}$ be (first-order) definable subset relation, so that $H_{\mathscr{R}_{\mathrm{def}}}(\alpha) = L_\alpha$. It is consistent that $2 \notin O_{\subseteq}$ (when 1 has undecidable subsets, i.e. $\alpha \subseteq 1$ with $\varnothing \in \alpha \vee \varnothing \notin \alpha$ failing to hold), but $2 = L_2 \in O_{\mathscr{R}_{\mathrm{def}}}$ always, so consistently $O_{\subseteq} \neq O_{\mathscr{R}_{\mathrm{def}}}$.

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- Using realisability arguments, we can verify

Theorem (W., upcoming PhD thesis)

IZF *is consistent with* $\alpha \neq \beta \wedge L_\alpha = L_\beta$.

Thus also consistently $O_{\mathscr{R}_{\mathrm{def}}} \neq \mathrm{Ord} = O_{\in}$.

Plump ordinals are useful!

It remains open from Lubarsky (1993) whether $\mathrm{IZF} \vdash \mathrm{Ord} \subseteq L$. However,

Theorem (W., 2026)

$O_{\subseteq} \subseteq L$. *More specifically, for any* $\alpha \in O_{\subseteq}$,

$$\alpha = \{\beta \in L_\alpha : L_\alpha \vDash \beta \in O_{\subseteq}\}.$$

(We need to phrase $O_{\subseteq}$ as Taylor's plump ordinals, to avoid mentioning the objects $H_{\subseteq}(\beta) = V_\beta$ in L_α.)

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Combined with the use of incomparable ordinals, it follows that

Theorem (W., 2026)

Fix any set x in any ZFC universe, then there is a Heyting algebra $\mathbb{H}$ such that the Heyting extension satisfies

$$V^{\mathbb{H}} \vDash \mathcal{P}(\check{x}) \in L.$$

Ordinal arithmetic

Fix any $\mathscr{R}$, we can let

$$\alpha +_{\mathscr{R}} \beta = \alpha \cup \bigcup_{\gamma\in\beta} (\alpha +_{\mathscr{R}} \gamma)^{\mathscr{R}+}, \quad \alpha \cdot_{\mathscr{R}} \beta = \bigcup_{\gamma\in\beta} (\alpha \cdot_{\mathscr{R}} \gamma +_{\mathscr{R}} \alpha).$$

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Lemma

For any $\alpha, \beta, \gamma \in \mathrm{Ord} = O_{\in}$, if $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$; if $\alpha + \beta \in \alpha + \gamma$, then $\beta \in \gamma$.

Lemma (W., 2026)

For any $\alpha, \beta, \gamma \in O_{\subseteq}$, if $\alpha +_{\subseteq} \beta = \alpha +_{\subseteq} \gamma$, then $\beta = \gamma$; if $\alpha +_{\subseteq} \beta \subseteq \alpha +_{\subseteq} \gamma$, then $\beta \subseteq \gamma$.

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For any $\alpha, \beta, \gamma \in O_{\subseteq}$, *if* $\alpha +_{\subseteq} \beta = \alpha +_{\subseteq} \gamma$, *then* $\beta = \gamma$; *if* $\alpha +_{\subseteq} \beta \subseteq \alpha +_{\subseteq} \gamma$, *then* $\beta \subseteq \gamma$.

These proofs are different (and indeed, the $O_{\subseteq}$ one is even simpler), and I do not know any uniform way to do this for arbitrary $\mathscr{R}$.

Thank you!

[1] Robert Seth Lubarsky, *Intuitionistic L*, Logical methods: In honor of Anil Nerode's sixtieth birthday (John Newsome Crossley, Jeffrey Brian Remmel, Richard Arnold Shore, and Moss Eisenberg Sweedler, eds.), Birkhäuser Boston, 1993, pp. 555–571.

[2] Paul Taylor, *Intuitionistic sets and ordinals*, The Journal of Symbolic Logic **61** (1996), no. 3, 705–744.

[3] Shuwei Wang, *Some notes on plump ordinals* (2026), available at `arXiv:2601.23070[math.LO]`.