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Classical and intuitionistic behaviours

of the constructive constructible universe

Constructive set theories

Let IKP denote Kripke–Platek set theory in intuitionistic logic, i.e. the theory with Δ_0-separation, Δ_0-collection and the unrestricted set induction scheme

$$\forall x\,(\forall y \in x\ \varphi(y) \to \varphi(x)) \to \forall x\,\varphi(x).$$

Let CZF^- denote IKP + unrestricted strong collection; let CZF denote CZF^- + subset collection

$$\begin{aligned}&\exists c\,\forall u\,(\forall x \in a\ \exists y \in b\ \varphi(x,y,u)\\ &\quad\to \exists d \in c\,(\forall x \in a\ \exists y \in d\ \psi(x,y,u) \wedge \forall y \in d\ \exists x \in a\ \psi(x,y,u)))\,.\end{aligned}$$

Let $\mathrm{IKP}(\mathcal{P})$ be the extension of IKP by the primitive powerset symbol; let $\mathrm{CZF}(\mathcal{P})$ simply be CZF + powerset.

The constructible universe

Let $\mathrm{Def}(x)$ be the collection of all definable subsets (with parameters) of $\langle x; \in \rangle$ in the language of set theory. We define L_α using one uniform clause

$$L_\alpha = \bigcup_{\beta \in \alpha} \mathrm{Def}(L_\beta).$$

Lubarsky (1993) verified well-definedness and basic properties of intuitionistic L (in IZF), but IKP already suffices for his arguments.

Alternatively, Matthews and Rathjen (2024) avoided the use of Def (and thus ω) and defined L through Barwise's *fundamental operations*. This construction works in IKP – infinity.

Non-classical ordinals

The proof of ordinal trichotomy

$$\forall \alpha, \beta \in \mathrm{Ord}\,(\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha)$$

invokes atomic excluded middle $\forall x, y\,(x \in y \vee x \notin y)$, and is easily violated in intuitionistic models (as we shall soon see).

Theorem (W. (2026), section 4.4)

1. CZF *is consistent with* $\exists \alpha \in \mathrm{Ord}\ \neg L_\alpha \cap \mathrm{Ord} = \alpha$;
2. CZF *is consistent with* $\exists \alpha, \beta \in \mathrm{Ord}\,(\alpha \neq \beta \wedge L_\alpha = L_\beta)$.

One biggest problem of whether $\mathrm{Ord} \subseteq L$ *is provable in intuitionistic logic* still remains open.

The realisability model

We explain the realisability model $V(\mathcal{A})$ from Rathjen (2006). Here $\mathcal{A}$ is a *partial combinatory algebra* (henceforce *pca*), i.e. any set with a partial binary operation $\cdot : \mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A}$, such that there exists distinguished objects $\mathbf{k}, \mathbf{s} \in \mathcal{A}$ satisfying

1. for $a, b \in \mathcal{A}$, $\mathbf{k} \cdot a \cdot b{\downarrow} = a$;
2. for $a, b, c \in \mathcal{A}$, $\mathbf{s} \cdot a \cdot b{\downarrow}$, and

$$a \cdot c \cdot (b \cdot c){\downarrow} \to \mathbf{s} \cdot a \cdot b \cdot c{\downarrow} = a \cdot c \cdot (b \cdot c).$$

(We follow Faber and van Oosten (2016); in other places this is called a *weak pca* and people use a stricter condition 2.)

One may also want a pairing function, i.e. $\mathbf{p}, \mathbf{p}_0, \mathbf{p}_1 \in \mathcal{A}$ satisfying

$$\mathbf{p}_0 \cdot (\mathbf{p} \cdot a \cdot b){\downarrow} = a, \quad \mathbf{p}_1 \cdot (\mathbf{p} \cdot a \cdot b){\downarrow} = b,$$

as well as recursive functions on natural numbers, e.g. succ.

The realisability model

We inductively define $V(\mathcal{A}) = \bigcup_{\alpha \in \mathrm{Ord}} V(\mathcal{A})_\alpha$ where

$$V(\mathcal{A})_\alpha = \bigcup_{\beta \in \alpha} \mathcal{P}\Big(\mathcal{A} \times V(\mathcal{A})_\beta\Big).$$

This can be constructed in any extension of IKP, where $\mathcal{P}(x)$ is a definable class, by building $V(\mathcal{A})$ as the Σ_1-inductive class $I(\Phi)$ where

$$\Phi = \{\langle x, y\rangle : x \subseteq \mathcal{A} \times y\}$$

(cf. Aczel and Rathjen (2010), section 19.3).

We (simultaneously) recursively define

$$\begin{aligned}
d \Vdash u \in v \quad &:\Leftrightarrow \quad \exists \langle e, x\rangle \in v \, (\mathbf{p}_0 d{\downarrow} = e \wedge \mathbf{p}_1 d{\downarrow} \Vdash u = x)\,, \\
d \Vdash u = v \quad &:\Leftrightarrow \quad \forall \langle e, x\rangle \in u \; \mathbf{p}_0 de{\downarrow} \Vdash x \in v \\
&\qquad\qquad \wedge \forall \langle e, x\rangle \in v \; \mathbf{p}_1 de{\downarrow} \Vdash x \in u.
\end{aligned}$$

The realisability model

Here are the rest of the realisability conditions:

$$
\begin{aligned}
d \Vdash \varphi \wedge \psi \quad &:\Leftrightarrow \quad \mathbf{p}_0 d{\downarrow} \Vdash \varphi \wedge \mathbf{p}_1 d{\downarrow} \Vdash \psi, \\
d \Vdash \varphi \vee \psi \quad &:\Leftrightarrow \quad (\mathbf{p}_0 d{\downarrow} = 0 \wedge \mathbf{p}_1 d{\downarrow} \Vdash \varphi) \\
&\qquad \vee (\mathbf{p}_0 d{\downarrow} = 1 \wedge \mathbf{p}_1 d{\downarrow} \Vdash \psi), \\
d \Vdash \neg\varphi \quad &:\Leftrightarrow \quad \forall e \in \mathcal{A}\ e \nVdash \varphi, \\
d \Vdash \varphi \to \psi \quad &:\Leftrightarrow \quad \forall e \in \mathcal{A}\,(e \Vdash \varphi \to de{\downarrow} \Vdash \psi), \\
d \Vdash \forall x \in a\ \varphi(x) \quad &:\Leftrightarrow \quad \forall \langle e, x\rangle \in a\ de{\downarrow} \Vdash \varphi(x), \\
d \Vdash \exists x \in a\ \varphi(x) \quad &:\Leftrightarrow \quad \exists \langle e, x\rangle \in a\,(\mathbf{p}_0 d{\downarrow} = e \wedge \mathbf{p}_1 d{\downarrow} \Vdash \varphi(x)), \\
d \Vdash \forall x\, \varphi(x) \quad &:\Leftrightarrow \quad \forall x \in V(\mathcal{A})\ d \Vdash \varphi(x), \\
d \Vdash \exists x\, \varphi(x) \quad &:\Leftrightarrow \quad \exists x \in V(\mathcal{A})\ d \Vdash \varphi(x).
\end{aligned}
$$

The realisability model

Theorem (Well-known)

For T being any of CZF^-, CZF or $\mathrm{CZF}(\mathcal{P})$, if φ is a theorem of T, then $T \vdash \exists d \in \mathcal{A}\ d \Vdash \varphi$.

The following are canonical names for the natural numbers and ω in the model:

$$\overline{n} = \{\langle m, \overline{m}\rangle : m < n\},$$
$$\overline{\omega} = \{\langle m, \overline{m}\rangle : m \in \omega\}.$$

However, some arbitrary $\{\langle a_m, \overline{m}\rangle : m \in \omega\} \in V(\mathcal{A})$ need not be a name for ω, depending on

1. whether each a_n are distinct, and
2. the extent to which $\mathcal{A}$ *cannot* compute $\{a_n\}_{n\in\omega}$.

Failure of classical-ness

Note that in any extension of IKP we have

Theorem (Lubarsky (1993))

L_ω satisfies atomic excluded middle, i.e.

$$\forall x, y \in L_\omega \, (x \in y \vee x \notin y) \, .$$

Proof sketch.

Check that the Ackermann bijection $f : L_\omega \to \omega$ is still well-defined in intuitionistic logic. □

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However, simply take Kleene's first algebra Kl,

Theorem (W. (2026), Proposition 7.1.4)

$V(Kl) \vDash L_{\omega+1}$ does not satisfy atomic excluded middle.

Proof sketch.

Let $\varphi(n)$ be the sentence which defines the halting set in $L_{\omega+1}$. Kl cannot decide whether $\varphi(n)$ is true for each $n \in \omega$, thus membership for the corresponding set cannot satisfy excluded middle. □

Linearity of ordinals

We say that an ordinal $\alpha \in \mathrm{Ord}$ is *linear* if it satisfies ordinal trichotomy

$$\forall \beta, \gamma \in \alpha \, (\beta \in \gamma \vee \beta = \gamma \vee \gamma \in \beta) \, .$$

Theorem (W. (2026), Theorem 7.1.6, IKP)

If $\alpha \in \mathrm{Ord}$ is linear, and $L_\alpha \cap \mathrm{Ord}$ is also linear, then L_α satisfies $\varphi \vee \neg\varphi$ for every Δ_0-formula φ.

Proof sketch.

Using the linearity of α, we replicate the classical proof that L_α satisfies Δ_0-separation. It follows that there is $\delta = \{x \in \{\varnothing\} : \varphi\} \in L_\alpha$ such that $\varnothing \in \delta \vee \delta = \varnothing$ holds. □

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Corollary

$V(Kl) \vDash L_{\omega+1} \cap \mathrm{Ord}$ *is not linear, thus* $L_{\omega+1} \cap \mathrm{Ord} \neq \omega + 1$.

Incomparable codings

In W. (2027), we (essentially) consider a family of names

$$\alpha = \{\langle a_m, \overline{m}\rangle : m \in \omega\} \in V(Kl),$$

where the functions $a_m \mapsto m$ and $a_{m+1} \mapsto a_m$ are recursive, but $m \mapsto a_m$ is not recursive. It follows that

Lemma
$V(Kl) \vDash \alpha \in \mathrm{Ord},\ \alpha \subseteq \omega,\ \omega \not\subseteq \alpha.$

By a priority argument, we can get a family of such ordinals $\{\alpha_n\}_{n\in\omega}$ such that additionally,

$$V(Kl) \vDash \alpha_i \not\subseteq \alpha_j \quad \text{for } i \neq j.$$

Incomparable codings

As $\forall n \in \omega\ L_n \cap \mathrm{Ord} = n$, we can show that each

$$L_{\alpha_n} \cap \mathrm{Ord} = \bigcup_{m \in \alpha_n} \mathrm{Def}(L_m) \cap \mathrm{Ord} = \bigcup_{m \in \alpha_n} (m+1) = \alpha_n.$$

Thus taking any $s \subseteq \omega$, it will be inter-definable with the ordinal

$$\delta = \bigcup_{n \in s} \mathrm{Def}(L_{\alpha_n}) \cap \mathrm{Ord} = L_{\bigcup_{n \in s}(\alpha_n+1)} \cap \mathrm{Ord} \in L$$

by $s = \{n \in \omega : \alpha_n \in \delta\}$.

Theorem (Variant of W. (2027), Theorem 12)

$V(Kl) \vDash \mathcal{P}(\omega) \subseteq L$.

α-recursion

We work in $\mathrm{CZF}^- + \mathrm{REM}$, i.e. with excluded middle $\varphi \vee \neg\varphi$ for every Δ_0-formula φ. This suffices to show that ordinals are linear and each L_α is classical.

Consider an *admissible* ordinal α, i.e. such that L_α satisfies Δ_0-collection. There is a notion of (partial) α-recursive functions (cf. Sacks (1990), Chapter VII), which are essentially Σ_1-definable partial class functions of the model $\langle L_\alpha; \in \rangle$, that forms a pca ΣL_α.

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For any $x \subseteq L_\alpha$ in the base theory, we can now recursively define

$$\overline{x} = \{\langle y, \overline{y} \rangle : y \in x\} \in V(\Sigma L_\alpha).$$

Question

Is $L_{\overline{\alpha}}$ (somewhat) classical in $V(\Sigma L_\alpha)$? In other words, does $L_{\overline{\alpha}} = \overline{L_\alpha}$ hold?

α-recursive realisability

Observe that $L_\alpha \vDash \mathrm{KP}$, and both the pca ΣL_α and the domain $\overline{L_\alpha}$ can be seen as definable classes of $\langle L_\alpha; \in \rangle$. The two objects produce a realisability model themselves.

Unrestricted collection is basically for free here:

Proof sketch for collection.
Suppose $a \in L_\alpha$ and $f \Vdash \forall x \in \overline{a}\ \exists y \in \overline{L_\alpha}\ \varphi^{\overline{L_\alpha}}(x, y)$, then for any $x \in a$, $\mathbf{p}_0(fx)\downarrow$ is precisely the witness needed for y. We just form the Σ_1-definable object

$$b = \{\mathbf{p}_0(fx) : x \in a\} \in L_\alpha,$$

and $\overline{b}$ is the desired result of collection in the realisability model. □

(For strong collection, we will need a choice principle to realise the converse, $\forall y \in \overline{b}\ \exists x \in \overline{a}\ \varphi^{\overline{L_\alpha}}(x, y)$; but L_α has a Σ_1-definable choice function by its global well-ordering.)

α-recursive realisability

Theorem (W. (2026), Theorem 7.3.8)

$V(\Sigma L_\alpha) \vDash \left(\overline{L_\alpha} \vDash \mathrm{CZF}^- + \mathrm{REM}\right)$.

For each $\beta \in \alpha$, we can already define $L_{\overline{\beta}}$ in $\overline{L_\alpha}$, which will be a classical object. By absoluteness of the L-hierarchy,

$$L_{\overline{\alpha}} = \bigcup_{\beta\in\alpha} \mathrm{Def}\left(L_{\overline{\beta}}\right) \subseteq \overline{L_\alpha}$$

and thus

Corollary

$V(\Sigma L_\alpha)$ *satisfies that for every* $\gamma \in \mathrm{Ord}$, *if* $\gamma \subseteq \overline{\alpha}$, *then* $L_\gamma \cap \mathrm{Ord} = \gamma$.

α-family of incomparable ordinals

By an α-recursive priority argument, we can build a family of names

$$\sigma_\gamma = \left\{ \left\langle a_{\beta,\gamma}, \overline{\beta} \right\rangle : \beta \in \alpha \right\} \in V(\Sigma L_\alpha)$$

for $\gamma \in \alpha$, such that:

Lemma (Inside $V(\Sigma L_\alpha)$)

1. *for any $\gamma \in \overline{\alpha}$, $\sigma_\gamma \subseteq \overline{\alpha}$ is a linear ordinal;*
2. *for any $\gamma, \delta \in \overline{\alpha}$, $\sigma_\gamma \subseteq \sigma_\delta \to \gamma = \delta$.*

Corollary (W. (2026), Corollary 7.3.16, $\mathrm{CZF}^- + \mathrm{REM}$)

For any ordinal β, if there exists an admissible ordinal α such that $\beta \subseteq \alpha$, then there exists a pca $\mathcal{A}$ such that $\beta \subseteq \mathcal{A}$ and

$$V(\mathcal{A}) \vDash \mathcal{P}\left(\overline{\beta}\right) \subseteq L.$$

Class-sized realisability

Question

Can we embed the entire $L = \bigcup_\alpha L_\alpha$ with a single model $V(\mathcal{A})$?

There are Σ_1-definable class-sized pcas, such as the set-recursive functions E, which are used in other realisability interpretations, e.g. Rathjen (2012a, 2012b).

$$\boldsymbol{\gamma} xy{\downarrow} = x \cap \left(\bigcap y\right),$$
$$\boldsymbol{\rho} xy{\downarrow} = \{xu : u \in y\},$$
$$\boldsymbol{\nu} x{\downarrow} = \bigcup x,$$
$$\boldsymbol{\pi} xy{\downarrow} = \{x, y\}.$$

$$\mathbf{i}_1 xyz{\downarrow} = \begin{cases} x & \text{when } y \in z, \\ \varnothing & \text{when } y \notin z, \end{cases}$$
$$\mathbf{i}_2 xyz{\downarrow} = \{u \in x : u \in y \to u \in z\},$$
$$\mathbf{i}_3 xyz{\downarrow} = \{u \in x : u \in y \vee z \in u\}.$$

Class-sized realisability

However, when $\mathcal{A}$ is class-sized, $V(\mathcal{A})$ can have many annoying caveats. This is discussed in W. (2026), section 7.2. Errata: I actually overlooked some technicalities there, so it is actually *more* problematic than claimed in my thesis.

1. To realise Δ_0-separation, we need a set-sized representation for possible realisers of an atomic formula $x \in y$. Therefore, to realise mere CZF^-, we need either the base theory CZF^- + exponentiation (which, when REM is added, implies the powerset axiom) or a global well-ordering.

Class-sized realisability

2. Even when the base theory has powerset, we cannot realise powerset. Indeed,

Theorem (W. (2026), Theorem 7.2.15)

$V(E)$ realises ¬powerset.

Proof sketch.
Suppose that $y \in V(E)$ is the "powerset" of $\overline{1}$. Observe that

$$\lambda x.0 \Vdash \{\langle a, \varnothing\rangle\} \subseteq \overline{1}$$

for class-many a, and there are not enough different indices to uniformly realise $\exists x \in y\ x = \{\langle a, \varnothing\rangle\}$ for all such a. □

3. Also, the previous argument for incomparables through non-computability become impossible, because E-recursion computes every set-sized function $f : a \to b$.

More incomparables?

However, $V(\mathcal{A})$ still supports many other forms of incomparable ordinals, e.g. for $u \subseteq \omega$,

$$\overline{\omega}_{(u)} = \{\langle n, \overline{n}\rangle : n \in \omega\} \cup \{\langle 0, \overline{n}\rangle : n \in u\} \in V(\mathcal{A}),$$

which instead satisfy $\overline{\omega} \subseteq \overline{\omega}_{(u)} \wedge \overline{\omega}_{(u)} \not\subseteq \overline{\omega}$, or more specifically

Lemma (W. (2026), Lemma 4.4.2)

For any $u, v \subseteq \omega$ such that $0 \in u, v$, we have $V(\mathcal{A}) \models \overline{\omega}_{(u)} \subseteq \overline{\omega}_{(v)}$ if and only if $u \subseteq v$.

But it will be up to further research whether these incomparables can be used to construct encodings or other interesting theories.

Thank you!

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