
**Higher-Order Conceptual Mathematics and
Intuitionistic Set Theory: Well-Orderings,
Ordinals and Gödel's L**

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others. The thesis incorporates material from solely-authored arXiv manuscripts [47–50], as indicated in the main text. [47] has additionally been published online as an open-access FirstView article in the *Journal of Symbolic Logic* at the time of submission.

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Abstract

This thesis studies the construction of realisability models from various partial combinatory algebras in formal theories of both arithmetic and set theory. The focus is on the representation of ‘set objects’ (or higher-order objects, respectively) and the implementation of transfinite recursion along well-orderings and set-theoretic ordinals for the purpose of interpreting specific axioms.

For the arithmetic part, we study a semi-intuitionistic theory of third-order conceptual mathematics, CM, proposed by Nik Weaver in [53]. We provide an ordinal analysis through a realisability interpretation in fragments of classical second-order arithmetic and use it to gauge the effect of adding different instances of the transfinite induction scheme on a global well-ordering to the proof-theoretic strength of the system.

The rest of the thesis then implements several realisability arguments over different partial combinatory algebras for constructive Zermelo–Fraenkel set theory. We obtain multiple consistency and independence results surrounding the intuitionistic ordinals and Gödel’s constructible universe L in this way. We also analyse and resolve the open problem of whether adding the axiom $V = L$ will increase the consistency strength of certain constructive set theories, giving a negative answer.

The set-theoretic contribution of this thesis is the method of incomparable codings, which utilises the existence of incomparable ordinals in intuitionistic set theory to encode information about arbitrary sets into a single ordinal. In conjunction with Paul Taylor’s notion of plump ordinals in [45], this leads to a very useful mechanism for verifying exotic properties of L in intuitionistic models.

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Chapter 1

Introduction

Realisability interpretation is a technique that dates back to Kleene [22] and serves as an important method for intuitionistic model construction. Compared to other approaches to the semantics of intuitionistic logic such as Kripke frames or Heyting-valued forcing, realisability arguments utilise a partial combinatory algebra to equip a theory with a constructive, often computational interpretation, and can provide key intuition as to why certain principles, such as dependent choice or the collection scheme in set theory, carry much less proof-theoretic strength when used in an intuitionistic context.

The main interest of this thesis is to examine different ways in which the construction of ‘set objects’, both in higher-order arithmetic and in set theories, can be carried out in a realisability model. We aim to study the degree to which various notions of computation — Turing machines and other topics in higher recursion theory: hyperarithmetic functions, α -recursion and set recursion — can interact with infinitary objects in a realisability model and support transfinite constructions along well-orderings and set-theoretic ordinals.

More specifically, we will look at various extensions and fragments of a semi-intuitionistic theory of third-order arithmetic, and of constructive Zermelo–Fraenkel set theory, CZF. We use a number of realisability constructions to interpret them in well-studied classical systems such as fragments of second-order arithmetic and extensions of Kripke–Platek set theory. As a consequence, we obtain many equiconsistency and independence results for different axioms in these contexts.

In chapter 2, we begin by introducing the formal definition for standard and weak partial combinatory algebras. We will present a general formulation

of the realisability conditions for logical connectives and quantifiers (which every realisability model in later chapters will inherit and modify slightly), and observe carefully that the rules of intuitionistic logic can be implemented in this system.

We shall also make an important distinction between *interpreted* realisability models, where the domain of the model remains a subset of the combinatory algebra, and *uninterpreted* realisability models, where entities in the model can be arbitrary constructs in the base theory and application terms only operate on partial information extracted from them. Many arguments in later chapters will exemplify and take advantage of their difference in behaviour.

In chapter 3, we will examine the third-order theory of conceptual mathematics, CM, proposed by Nik Weaver in [53] as an alternative foundation for mainstream mathematics. An ordinal analysis is performed through a realisability argument to determine that:

Corollary 3.2.9. $\text{CM} \equiv_{\text{Con}} \Sigma_1^1\text{-DC}$, and we have $|\text{CM}| = |\Sigma_1^1\text{-DC}| = \varphi_{\varepsilon_0}(0)$.

Additionally, we give a detailed discussion of the extension of CM by a global well-ordering axiom that Weaver mentioned briefly in [53, section 2.3]. We will extend our realisability interpretation to show that an unrestricted transfinite induction scheme easily raises the proof-theoretic strength of the extension to the Bachmann–Howard ordinal, $\vartheta_{\varepsilon_{\Omega+1}}(0)$. A weaker, predicatively flavoured restriction will be presented, along with some consistency strength bounds for other fragments of this theory of a global well-ordering.

In chapter 4, we then turn towards the study of intuitionistic set theories. We will present the formal statements of the axioms of a hierarchy of set theories: IKP, CZF, CZF($\mathcal{P}$) and IZF. In particular, we will investigate CZF and the standard realisability interpretation of this theory in itself by Rathjen [31].

The chapter also covers the construction of Gödel’s constructible universe L in an intuitionistic context, and its various exotic behaviours in the given realisability model. We will construct specific examples to conclude that:

Corollary 4.4.11. CZF is consistent with $\exists \alpha \in \text{Ord} \neg L_\alpha \cap \text{Ord} = \alpha$.

Corollary 4.4.13. CZF is consistent with $\exists \alpha, \beta \in \text{Ord} (\alpha \neq \beta \wedge L_\alpha = L_\beta)$.

In chapter 5, we will develop a method of *incomparable codings* to make use of these intuitionistic ordinals that are incomparable under the $\in$ - or $\subseteq$ -relation. We show that, given the existence of sufficiently many such incomparable ob-

jects, one may encode information about the internal structure of an arbitrary set into a single ordinal, and thus implement scenarios where a highly ‘non-constructive’ set can be found inside L . We cover the key result from my manuscript [47] that:

Theorem 5.1.7 (IKP). *Assume there exist weakly incomparable $\alpha, \beta \in \text{Ord}$, then the following are equivalent:*

- (i) $\text{Ord} \subseteq L$,
- (ii) $V = L$.

As another application of the technique, we will also work with a variant of Aczel’s realisability interpretation of CZF in Martin-Löf type theory [2] and use incomparable codings to prove $V = L$ inside the model. This allows us to justify the following equiconsistency result, where the classical, trivial proof of the similar fact in ZF fails to apply (cf. section 5.2 for more detail):

Corollary 5.4.7. $\text{CZF} \equiv_{\text{Con}} \text{CZF} + V = L$.

In chapter 6, we give a formal treatment of Paul Taylor’s notion of plump ordinals in [45], and demonstrate how it leads to a semi-automatic process for constructing incomparable coding objects. We will combine this result with the weak realisability interpretation of $\text{CZF}(\mathcal{P})$ in a classical base theory $\text{KP}(\mathcal{P})$ in [32] to extend the construction above to a model satisfying the powerset axiom, and show that $\text{CZF}(\mathcal{P})$ also remains equiconsistent with its extension $\text{CZF}(\mathcal{P}) + V = L$.

In chapter 7, finally, we will discuss how to embed a larger classical fragment into an intuitionistic realisability universe via the use of large combinatory algebras including the α -recursive and set recursive functions. We will explore how such realisability models may support a direct construction of large pairwise incomparable functions, unlike the hierarchy method employed in chapter 6, and use this to prove a realisability-based analogue of an existing result of mine in [50, section 5] for Heyting-valued forcing:

Theorem 7.3.16 ($\text{CZF}^- + \text{REM}$). *For any ordinal β , if there exists an admissible ordinal α such that $\beta \subseteq \alpha$, then there exists a (weak) pca $\mathcal{A}$ such that $\beta \subseteq \mathcal{A}$ and the corresponding realisability model*

$$V(\mathcal{A}) \models \mathcal{P}(\check{\beta}) \subseteq L.$$

Chapter 2

Partial combinatory algebras and realisability

To begin this thesis, we will formally define the notions of a *partial combinatory algebra* (henceforth abbreviated as *pca*) and realisability models. Most things covered in this chapter should be well-known folklore (cf. Beeson [5, Chapters VI & VII]), but we will give a detailed account here with the aim of presenting a self-contained exposition on the realisability arguments employed in the thesis.

Additionally, we observe that the notations for application terms vary a lot in existing literature, and this chapter shall serve as the notational convention to be followed in the rest of the chapters. In section 2.3, we also present a Hilbert-style system similar to Kleene’s original formulation [23], which should be understood as the precise meaning of ‘first-order intuitionistic logic’ throughout this thesis.

2.1 Partial combinatory algebras

A *partial combinatory algebra* (*pca*) is a set¹ $\mathcal{A}$ together with a partial function $\text{ap} : \mathcal{A}^2 \rightharpoonup \mathcal{A}$ denoting the *function application* operation and two distinguished objects $\mathbf{k}, \mathbf{s} \in \mathcal{A}$ satisfying their respective defining equations.

Before laying out these equations, we first stipulate that, since function application is the sole core instrumentation in all our realisability models, we will reserve the juxtaposition notation for it, i.e. we will simply write ab to denote

¹Here we develop some general theory about pcas without fixing a particular formal base theory, so we use the word ‘set’ in an informal sense, and we actually allow the domain $\mathcal{A}$ to be a proper definable class in some theories as well.

$\text{ap}(a, b)$. Numerical multiplication, instead, will always be denoted with a dot as $a \cdot b$ in this thesis.

Following conventions, juxtaposition here shall associate to the left, i.e. the expression abc should be read as $(ab)c$ or $\text{ap}(\text{ap}(a, b), c)$. We also write $ab \downarrow$ and say that the application *converges* to indicate that $ab = \text{ap}(a, b) \in \mathcal{A}$ is defined; we say the application *diverges* otherwise.

Now, the axioms of a pca shall enforce that:

- (i) for any $a, b \in \mathcal{A}$, $\mathbf{k}a \downarrow$, $\mathbf{k}ab \downarrow$, and

$$\mathbf{k}ab = a;$$

- (ii) for any $a, b, c \in \mathcal{A}$, $\mathbf{s}a \downarrow$, $\mathbf{s}ab \downarrow$, and²

$$\mathbf{s}abc = ac(bc).$$

In practice, we will work with objects in a pca through λ -abstraction. We say that the *formal application terms* of $\mathcal{A}$ over a finite enumeration of variables $x_1, \dots, x_k$ are defined inductively as follows:

- (1) any variable x_i or any element in $\mathcal{A}$ is a formal application term;
- (2) if t, s are formal application terms, then (ts) is a formal application term;
- (3) nothing else is a formal application term.

We shall abuse notation and identify any closed formal application term t (i.e. a formal application term in which no variables occur) with the result of its computation, i.e. either an element in $\mathcal{A}$ or undefined, and write that $t \downarrow$ if the computation converges. For any formal application term t over variables $x_1, \dots, x_k$ and any $a_1, \dots, a_k \in \mathcal{A}$, the *evaluation* $t[a_1/x_1, \dots, a_k/x_k]$ is then the result of substituting each x_i by the corresponding a_i . We verify that:

Lemma 2.1.1 (λ -abstraction). *For any formal application term t over variables $x, y_1, \dots, y_k$, there is a formal application term, henceforth denoted as $\lambda x. t$, over variables $y_1, \dots, y_k$ only, such that*

²When reasoning about partial combinatory algebras, we follow usual convention and allow one to write potentially undefined terms like $ac(bc)$ without requiring that $ac \downarrow$ or $bc \downarrow$ in the first place; we consider the term $ac(bc)$ to diverge if either ac or bc already diverges.

Additionally, for simplicity we will abuse the equality symbol between two application terms in this thesis, instead of introducing a new symbol $\simeq$ as in some other literature; when we write $t = s$ where t and s are potentially undefined application terms, this refers to the assertion that either t and s both diverge, or that they both converge and are equal.

- (i) for any $b_1, \dots, b_k \in \mathcal{A}$, $(\lambda x. t)[b_1/y_1, \dots, b_k/y_k] \downarrow$;
- (ii) for any $a, b_1, \dots, b_k \in \mathcal{A}$,

$$(\lambda x. t)[b_1/y_1, \dots, b_k/y_k]a = t[a/x, b_1/y_1, \dots, b_k/y_k].$$

Proof. We prove this by induction on the complexity of t . If $t = x$, take $\lambda x. t = \mathbf{skk}$; if $t = y_i$ or $t = a \in \mathcal{A}$, simply take $\lambda x. t = \mathbf{kt}$.

If $t = (rs)$ where r, s are formal application terms, by the inductive hypothesis we then have formal application terms $\lambda x. r$ and $\lambda x. s$. Taking $\lambda x. t = \mathbf{s}(\lambda x. r)(\lambda x. s)$ would suffice. $\square$

As an immediate consequence, any formal application term of the form $\lambda x. t$ where t has no variables other than x (including the case where t contains additional λ -terms which bind more variables) is a closed formal application term which *must* converge, i.e. $\lambda x. t$ is a definite element in $\mathcal{A}$.

In what follows, we will abbreviate $\lambda x_1. \lambda x_2. \dots \lambda x_k. t$ as $\lambda x_1 x_2 \dots x_k. t$, or simply as $\lambda \vec{x}. t$ where $\vec{x} = x_1, \dots, x_k$ denotes a sequence of variables; correspondingly, we will write $ts_1 \dots s_k$ as $t\vec{s}$ where $\vec{s} = \langle s_1, \dots, s_k \rangle$ is a tuple of (formal) application terms.

Corollary 2.1.2 (Recursion theorem). *There must be a distinguished element $\mathbf{r} \in \mathcal{A}$ such that for any $f, a \in \mathcal{A}$, $\mathbf{r}f \downarrow$ and*

$$\mathbf{r}fa = fa(\mathbf{r}f).$$

Proof. We build the well-known ‘Y-combinator’. Let $t = \lambda fga. fa(gg)$, and observe that

$$tf(tf)a = fa(tf(tf)).$$

Therefore, simply taking $\mathbf{r} = \lambda f. tf(tf)$ would suffice. Note that $\mathbf{r}f \downarrow$ is ensured because for any $f, g \in \mathcal{A}$, we have

$$tf g = (\lambda a. fa(gg)) \downarrow$$

by Lemma 2.1.1. $\square$

Pairing functions

We shall maintain the convention that every pca $\mathcal{A}$ is equipped with a distinguished *pairing function*, i.e. some $\mathbf{p} \in \mathcal{A}$ together with unpairing operations

$\mathbf{p}_0, \mathbf{p}_1 \in \mathcal{A}$ such that for any $a, b \in \mathcal{A}$,

$$\mathbf{p}_0(\mathbf{p}ab) = a \quad \text{and} \quad \mathbf{p}_1(\mathbf{p}ab) = b.$$

We do not require that the same (uniformly-defined) pairing function is used in all the pcas considered in this thesis, since in some scenarios we may want our pairing function to possess additional properties — for example, when the domain $\mathcal{A} = \mathbb{N}$ (cf. the next section for pcas over natural numbers), we may want the pairing function to be a bijection $\mathbb{N}^2 \rightarrow \mathbb{N}$, so that $\mathbf{p}_0, \mathbf{p}_1$ will also be total operations. However, do note that some default construction will work for an arbitrary pca, such as the following one mentioned in [17] (which is a typical Church encoding):

Lemma 2.1.3 ([17, footnote 5]). *Let $\mathbf{p} = \lambda abc. cab$, $\mathbf{p}_0 = \lambda p. p(\lambda ab. a)$ and $\mathbf{p}_1 = \lambda p. p(\lambda ab. b)$. Then $\mathbf{p}, \mathbf{p}_0, \mathbf{p}_1$ give a pairing function.*

Proof. Trivial. □

For convenience, we will also adopt the notation $\mathbf{p}_{i_0 i_1 \dots i_n}$ where each $i_k \in \{0, 1\}$ as the sequential composition of the unpairing operations, i.e.

$$\mathbf{p}_{i_0 i_1 \dots i_n} = \lambda x. \mathbf{p}_{i_n}(\mathbf{p}_{i_{n-1}} \cdots (\mathbf{p}_{i_0} x) \cdots).$$

Take note of the reversal of indices as I find this notation more intuitive to work with.

2.2 Natural numbers in pcas

We shall follow the terminology of [17] and use ‘pca over natural numbers’ to refer to a pca with the infrastructure to represent natural numbers and recursive functions on them. This thesis will not concern itself with a full formal theory of such structures, but rather views them as mere machinery within an established base theory to facilitate realisability constructions. Therefore, we directly say that a *pca over natural numbers* is a pca $\mathcal{A}$ with additional distinguished objects $\mathbf{s}_{\mathbb{N}}, \mathbf{p}_{\mathbb{N}}, \mathbf{d}_{\mathbb{N}} \in \mathcal{A}$ satisfying that:

- (i) the natural numbers (in the base theory) $\mathbb{N} \subseteq \mathcal{A}$;
- (ii) for any $n \in \mathbb{N}$, $\mathbf{s}_{\mathbb{N}}n = n + 1$ and³ $\mathbf{p}_{\mathbb{N}}(\mathbf{s}_{\mathbb{N}}n) = n$;

³Some sources would additionally require that $\mathbf{p}_{\mathbb{N}}0\downarrow = 0$. We do not demand this here, since one can always use $\mathbf{d}_{\mathbb{N}}$ in the next clause to test for zero first before applying $\mathbf{p}_{\mathbb{N}}$ safely.

(iii) for any $n, m \in \mathbb{N}$, $a, b \in \mathcal{A}$,

$$\mathbf{d}_{\mathbb{N}}nmab = \begin{cases} a & \text{if } n = m, \\ b & \text{if } n \neq m. \end{cases}$$

There is a subtlety to note here that, given our definition of term application on a pca, a term would only converge when all of its subterms converge, even when it is the case operation $\mathbf{d}_{\mathbb{N}}nmab$ that should always disregard one of its last two arguments. In order to have a ‘lazy’ alternative that does not require the convergence of all its subterms, we will enforce the abbreviation that

$$\mathbf{d}_{\mathbb{N}}^*nmab = \mathbf{d}_{\mathbb{N}}nm(\lambda z. a)(\lambda z. b)0$$

throughout this thesis, where z is a distinct new variable and a, b can be complex (formal) application terms so that $\lambda z. a$, $\lambda z. b$ are implemented according to Lemma 2.1.1, and $\mathbf{d}_{\mathbb{N}}nm(\lambda z. a)(\lambda z. b)$ is guaranteed to converge. Note that $\mathbf{d}_{\mathbb{N}}^*$ itself cannot be an actual element in $\mathcal{A}$, due to precisely the evaluation rule mentioned above.

Now, it is easy to verify that this pca $\mathcal{A}$ must contain a name for every partial recursive function on natural numbers, in the following sense:

Lemma 2.2.1. *For any partial recursive function $f : \mathbb{N}^k \rightarrow \mathbb{N}$, there is some $e \in \mathcal{A}$ such that for any tuple $\vec{n} = \langle n_1, \dots, n_k \rangle \in \mathbb{N}^k$,*

$$f(\vec{n})\downarrow \rightarrow e\vec{n}\downarrow = f(\vec{n}).$$

Proof. We use the standard characterisation that the collection of partial recursive functions is the closure of the successor, constant and projection functions under composition, primitive recursion and minimisation (cf. [44, Chapter I]). Now, given any $g, h \in \mathcal{A}$, primitive recursion

$$fx\vec{y} = \begin{cases} g\vec{y} & \text{if } x = 0, \\ h(\mathbf{p}_{\mathbb{N}}x)(f(\mathbf{p}_{\mathbb{N}}x)\vec{y})\vec{y} & \text{if } x > 0 \end{cases}$$

is given simply by $f = \mathbf{r}(\lambda xt\vec{y}. \mathbf{d}_{\mathbb{N}}^*x0(g\vec{y})(h(\mathbf{p}_{\mathbb{N}}x)(t(\mathbf{p}_{\mathbb{N}}x)\vec{y})\vec{y}))$; minimisation

$$f\vec{y} = \min \{x \in \mathbb{N} : gx\vec{y}\downarrow = 0 \wedge \forall z < x \, gz\vec{y}\downarrow\}$$

(if such x exists) is then $f = \lambda\vec{y}. \mathbf{r}(\lambda xt. \mathbf{d}_{\mathbb{N}}^*(gx\vec{y})0x(t(\mathbf{s}_{\mathbb{N}}x)))0$, where we recurse upwards (as long as $gx\vec{y}\downarrow$) until the first x such that $gx\vec{y} = 0$. $\square$

Indeed, consider the application function $\text{ap} : \mathbb{N}^2 \rightarrow \mathbb{N}$ given by Turing machine computation, i.e. $ea = b$ if the Turing machine with index e halts on input a and produces output b , then it is easy to observe that the distinguished operations $\mathbf{k}, \mathbf{s}, \mathbf{s}_{\mathbb{N}}, \mathbf{p}_{\mathbb{N}}, \mathbf{d}_{\mathbb{N}}$ are all themselves recursive (for $\mathbf{s}$ by the existence of the universal machine). This gives the standard pca on domain $\mathbb{N}$ known as *Kleene's first algebra*, which we denote as Kl .

2.3 Realisability of intuitionistic logic

Given a first-order language $\mathcal{L}$ and a fixed domain U , in this thesis we will work with terms/formulae together with some parameters from U as formal objects in the base theory.

A *term with parameters* is a term $t \in \mathcal{L}$ together with a finite tuple of variable assignments $a_1/x_1, \dots, a_k/x_k$ where each $a_i \in U$. This will be presented using the substitution notation $t[a_1/x_1, \dots, a_k/x_k]$ and, whenever there is no ambiguity, we may abuse notation and write, for example, simply $f(a)$ instead of $f(x)[a/x]$. Likewise, a *formula with parameters* is a formula $\varphi \in \mathcal{L}$ together with a finite tuple of variable assignments $a_1/x_1, \dots, a_k/x_k$ where each $a_i \in U$, denoted as $\varphi[a_1/x_1, \dots, a_k/x_k]$; and we will sometimes abuse notation and write $\varphi(a)$ instead.

We say that a term/formula with parameters is *closed* if all free variables of the expression appear in the accompanying variable assignments; otherwise it is *open* and its *free variables* are those that lack assignments.

Given a first-order language $\mathcal{L}$, an *interpretation* of $\mathcal{L}$ -terms (with parameters) in a pca $\mathcal{A}$ consists of a domain $U \subseteq \mathcal{A}$ together with, for each k -ary function symbol f in $\mathcal{L}$, some $f^{\mathcal{A}} \in \mathcal{A}$ such that for any $\vec{a} \in U^k$, $f^{\mathcal{A}}\vec{a} \downarrow \in U$. The interpretation clearly extends to all $\mathcal{L}$ -terms, where for each closed term t we correlate some $t^{\mathcal{A}} \in U$; and for each term t with (an explicit, complete enumeration of) free variables $x_1, \dots, x_k$, we have a function $t^{\mathcal{A}} \in \mathcal{A}$ such that for any $\vec{a} = \langle a_1, \dots, a_k \rangle \in U^k$,

$$t^{\mathcal{A}}\vec{a} \downarrow = t[a_1/x_1, \dots, a_k/x_k]^{\mathcal{A}}.$$

Given some *interpretation* of $\mathcal{L}$ -terms in a pca $\mathcal{A}$ over natural numbers⁴, a

⁴Note that the sole need for a pca over natural numbers here is for the disjunction clause, where one builds a disjoint union using 0 and 1. This can be avoided through a Church encoding of disjoint unions instead, for example, by requiring that $d \Vdash \varphi \vee \psi$ if and only if

collection⁵ of *realisability conditions* then consists of pairs $\langle d, \varphi \rangle$ where $d \in \mathcal{A}$ and φ is a closed $\mathcal{L}$ -formula (with parameters), where we say that ‘ d realises φ ’ and write $d \Vdash \varphi$ if $\langle d, \varphi \rangle$ is in the collection.

How atomic φ ’s are realised is usually specified on a case-by-case basis, depending on whether the truth conditions of the atomic formulae in our desired model are computable by the pca; however, most realisability models would extend its atomic conditions to the collection of all formulae through the following inductive clauses:

$$\begin{aligned}
d \Vdash \varphi \wedge \psi & :\iff \mathbf{p}_0 d \downarrow \Vdash \varphi \wedge \mathbf{p}_1 d \downarrow \Vdash \psi, \\
d \Vdash \varphi \vee \psi & :\iff (\mathbf{p}_0 d = 0 \wedge \mathbf{p}_1 d \downarrow \Vdash \varphi) \vee (\mathbf{p}_0 d = 1 \wedge \mathbf{p}_1 d \downarrow \Vdash \psi), \\
d \Vdash \neg \varphi & :\iff \forall e \in \mathcal{A} \neg e \Vdash \varphi, \\
d \Vdash \varphi \rightarrow \psi & :\iff \forall e \in \mathcal{A} (e \Vdash \varphi \rightarrow de \downarrow \Vdash \psi), \\
d \Vdash \forall x \varphi & :\iff \forall a \in U da \downarrow \Vdash \varphi[a/x], \\
d \Vdash \exists x \varphi & :\iff \mathbf{p}_0 d \downarrow \in U \wedge \mathbf{p}_1 d \downarrow \Vdash \varphi[\mathbf{p}_0 d/x].
\end{aligned}$$

To align better with the usual notion of proofs in first-order logic (where having open formulae as rows in a proof is crucial to many systems), we shall further extend the notion to open formulae: for any formula (with parameters) $\varphi \in \mathcal{L}$ with its free variables enumerated as $x_1, \dots, x_k$, we say that $d \in \mathcal{A}$ realises φ if and only if it realises the universal closure $\forall x_1, \dots, x_k \varphi$, i.e. if and only if for any $\vec{a} = \langle a_1, \dots, a_k \rangle \in U^k$,

$$d\vec{a} \Vdash \varphi[a_1/x_1, \dots, a_k/x_k].$$

Obviously the order of the variable enumeration does not affect the existence of a realiser, nor does the inclusion of additional free variables that do not actually occur in φ in the enumeration.

We can now fix a formal, intuitionistic proof system of first-order logic, for example the following Hilbert-style system that was (roughly) originally used by Kleene [23, Chapter IV] (cf. [27] and [46, Chapter 2]) with the axioms

$$(1) \quad \varphi \rightarrow (\psi \rightarrow \varphi),$$

either $\mathbf{p}_0 d \downarrow \Vdash \varphi$ and for any $f, g \in \mathcal{A}$, $\mathbf{p}_1 dfg = f(\mathbf{p}_0 d)$, or $\mathbf{p}_0 d \downarrow \Vdash \psi$ and for any $f, g \in \mathcal{A}$, $\mathbf{p}_1 dfg = g(\mathbf{p}_0 d)$ (so that the realiser d could be constructed as either $\mathbf{p}e(\lambda fg. fe)$ for $e \Vdash \varphi$ or $\mathbf{p}e(\lambda fg. ge)$ for $e \Vdash \psi$). But we opt for the simpler, conventional approach using natural numbers in this thesis.

⁵There is a caveat that sometimes, this collection will not be internally definable inside the base theory. See the discussion on page 12 for more detail.

-
- (2) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow \sigma)),$
 - (3) $\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi),$
 - (4) $\varphi \wedge \psi \rightarrow \varphi,$
 - (5) $\varphi \wedge \psi \rightarrow \psi,$
 - (6) $\varphi \rightarrow \varphi \vee \psi,$
 - (7) $\psi \rightarrow \varphi \vee \psi,$
 - (8) $(\varphi \rightarrow \sigma) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\varphi \vee \psi \rightarrow \sigma)),$
 - (9) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi),$
 - (10) $\neg\varphi \rightarrow (\varphi \rightarrow \psi),$
 - (11) $\forall x \varphi \rightarrow \varphi[t/x]$ where t is any term which is free⁶ for x in φ ,
 - (12) $\varphi[t/x] \rightarrow \exists x \varphi$ where t is any term which is free for x in φ ,

for any formulae φ , ψ and σ , together with the following inference rules:

- (i) *modus ponens*: from φ and $\varphi \rightarrow \psi$, deduce ψ ;
- (ii) $\forall$ -*introduction*: from $\varphi \rightarrow \psi$ where x does not occur free in φ , deduce $\varphi \rightarrow \forall x \psi$;
- (iii) $\exists$ -*elimination*: from $\varphi \rightarrow \psi$ where x does not occur free in ψ , deduce $\exists x \varphi \rightarrow \psi$.

Proposition 2.3.1 (Soundness of intuitionistic logic in realisability models). *Any logical axiom of the intuitionistic system is realised in any realisability model (of the form above). Moreover, for any inference rule of the system, if all antecedents are realised, then so is the consequent.*

Proof. Given any logical axiom (which is potentially an open formula), let $\vec{x}$ enumerate the free variables in it. Then it is easy to observe that:

- (1) $\lambda \vec{x} d e. d$ realises $\varphi \rightarrow (\psi \rightarrow \varphi)$;
- (2) $\lambda \vec{x} f g d. g d(f d)$ realises $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow \sigma))$;
- (3) $\lambda \vec{x}. \mathbf{p}$ realises $\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$;
- (4) $\lambda \vec{x} d. \mathbf{p}_0 d$ realises $\varphi \wedge \psi \rightarrow \varphi$;
- (5) $\lambda \vec{x} d. \mathbf{p}_1 d$ realises $\varphi \wedge \psi \rightarrow \psi$;

⁶In other words, no free variables in t will become bound when substituted for x in φ .

- (6) $\lambda \vec{x}d. \mathbf{p}0d$ realises $\varphi \rightarrow \varphi \vee \psi$;
- (7) $\lambda \vec{x}d. \mathbf{p}1d$ realises $\psi \rightarrow \varphi \vee \psi$;
- (8) $\lambda \vec{x}fgd. \mathbf{d}_{\mathbb{N}}(\mathbf{p}0d)0fg(\mathbf{p}1d)$, whose inner term evaluates to $f(\mathbf{p}1d)$ when the first disjunct of $\varphi \vee \psi$ is realised and to $g(\mathbf{p}1d)$ otherwise, realises $(\varphi \rightarrow \sigma) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\varphi \vee \psi \rightarrow \sigma))$;
- (9) $\lambda \vec{x}fg.0$ realises $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)$, since suppose otherwise⁷, i.e. $f \Vdash \varphi \rightarrow \psi$, $g \Vdash \varphi \rightarrow \neg\psi$, yet some $d \Vdash \varphi$, then $fd \Vdash \psi$ and $gd \Vdash \neg\psi$, contradicting the stipulation that $\neg\psi$ is realised only when no realisers of ψ exist;
- (10) $\lambda \vec{x}f.0$ realises $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$, since the existence of $f \Vdash \neg\varphi$ and $g \Vdash \varphi$ again implies a contradiction. Thus $f \Vdash \neg\varphi$, $g \Vdash \varphi$ entails $0g\downarrow \Vdash \psi$ as *ex falso quodlibet*.

For the quantifier axioms, assume that $t^A \in \mathcal{A}$ takes the same enumeration of arguments $\vec{x}$ above as input, then

- (11) $\lambda \vec{x}f. f(t^A\vec{x})$ realises $\forall x \varphi \rightarrow \varphi[t/x]$;
- (12) $\lambda \vec{x}d. \mathbf{p}(t^A\vec{x})d$ realises $\varphi[t/x] \rightarrow \exists x \varphi$.

For the inference rules, check that:

- (i) Assume that $d \Vdash \varphi$ and $f \Vdash \varphi \rightarrow \psi$, which (without loss of generality) both involve the same enumeration of free variables $\vec{x}$, then $\lambda \vec{x}. f\vec{x}(d\vec{x})$ realises ψ ;
- (ii) Assume $f \Vdash \varphi \rightarrow \psi$ where the free variables are enumerated as $x, \vec{y}$ and x does not occur free in φ , then $\lambda \vec{y}dx. f\vec{y}d$ realises $\varphi \rightarrow \forall x \psi$;
- (iii) Assume $f \Vdash \varphi \rightarrow \psi$ where the free variables are enumerated as $x, \vec{y}$ and x does not occur free in ψ , then $\lambda \vec{y}d. f(\mathbf{p}0d)\vec{y}(\mathbf{p}1d)$ realises $\exists x \varphi \rightarrow \psi$. $\square$

It follows immediately, by induction on the length of a proof, that given any intuitionistic theory T and a realisability model that realises all axioms of T , any theorem of T is realised as well.

Now, there is an **important** caveat concerning the formalisation of this argument: depending on the choice of a base theory and the pca $\mathcal{A}$ — especially when working with a ‘tight’ interpretation between two theories of the same

⁷This appears to be a proof-by-contradiction argument, but it applies in an intuitionistic base theory as well, since the statement to be proved, $\forall e \in \mathcal{A} \neg e \Vdash \varphi$, is itself a negated statement, thus it suffices to simply derive a contradiction.

proof-theoretic strength — the inductively-formulated collection of all realisability conditions may not even form a definable class in the base theory. So we actually have two distinct kinds of formal realisability arguments:

- (a) When the collection $\{\langle d, \varphi \rangle : d \Vdash \varphi\}$ of all realisability conditions is definable inside some base theory S , then the aforementioned induction on the length of a proof can also be carried out inside S , and thus the realisability argument simply yields a proof of $S \vdash \text{Con } T$ since no $d \in \mathcal{A}$ can realise $\perp$ by stipulation.
- (b) Alternatively, when the said collection cannot be defined internally, we could instead formulate a metatheoretic family of collections

$$r_\varphi = \{ \langle d, \overline{\varphi}[\vec{a}/\vec{x}] \rangle : \vec{a} \in U^k, d \Vdash \overline{\varphi}[\vec{a}/\vec{x}] \},$$

each of which alone is definable inside the theory S , where $\overline{\varphi}$ is the S -term denoting the ‘standard’ formula φ in the metatheory, for example a numerical constant representing the Gödel numbering of φ . It then follows that $r_\perp = \emptyset$, but the induction on proof length must be performed in the metatheory, resulting in an argument for $\vdash \text{Con } S \rightarrow \text{Con } T$ there.

The latter option is equivalent to the traditional interpretation-by-*translation* approach (cf. [5, Chapter VII]) which maintains that there is a metatheoretic function mapping each formula φ in the language of theory T to a formula in the language of S with one additional free variable d that captures the denotation of ‘ d realises φ ’. One ensures that proofs in intuitionistic logic are preserved and $\perp$ is unchanged under the translation, thus verifying that the consistency of S implies that of T .

In this thesis, however, I prefer the current presentation as I want to put more emphasis on the objects in the chosen domain U inside the base theory and the fact that we can still operate on these intuitionistic constructs (and speak about which formulae in the language of T ‘hold’ on them) in a very model-theoretic way, even in the second scenario. This intuition shall become very relevant in the later chapters on intuitionistic set theory.

Equality axioms

Across the variety of formal theories considered in this thesis, we allow some freedom in treating the equality symbol as either a primitive, logical relation symbol or a defined abbreviation (usually through extensional equality for sets)

— or, in the case of higher-order languages, as a primitive symbol for first-order variables, but as abbreviations for variables of higher orders, cf. section 3.1.

In the case where equality is primitive, we will need the following additional logical axioms:

- (E1) reflexivity, $x = x$;
- (E2) symmetry, $x = y \rightarrow y = x$;
- (E3) transitivity, $x = y \rightarrow (y = z \rightarrow x = z)$;
- (E4) the substitution scheme, $x = y \rightarrow (\varphi(x) \rightarrow \varphi(y))$ for any atomic formula $\varphi(x)$.

Since each realisability model will provide its own interpretation of a primitive equality symbol, the equality axioms need to be verified on a case-by-case basis. However, in several realisability interpretations we see a common pattern of enforcing

$$d \Vdash x = y \quad :\Longleftrightarrow \quad x = y$$

for any $d \in \mathcal{A}$ and $x, y \in U$, in which case we say that equality is *realised trivially*. We then have the following and do not have to worry about the equality axioms further:

Lemma 2.3.2. *If a realisability model realises equality trivially, then all the equality axioms are immediately realised.*

Proof. We simply check that:

- (E1) $\lambda x. 0$ realises $x = x$;
- (E2) $\lambda xy d. 0$ realises $x = y \rightarrow y = x$;
- (E3) $\lambda xyz d e. 0$ realises $x = y \rightarrow (y = z \rightarrow x = z)$;
- (E4) $\lambda xy \vec{z} d e. e$ realises $x = y \rightarrow (\varphi(x) \rightarrow \varphi(y))$, where $\vec{z}$ enumerates the remaining free variables in $\varphi(x)$, since for any $a, b \in U$, $\vec{c} \in U^k$, if $a = b$, then $\varphi(x)[a/x, \vec{c}/\vec{z}]$ and $\varphi(y)[b/y, \vec{c}/\vec{z}]$ are simply the same formula (with same parameters). $\square$

2.4 Uninterpreted realisability models

In other variants of realisability arguments, we often wish to choose a domain U that is not restricted by the size of the involved pca $\mathcal{A}$, for example, when we

want to build an uncountable set-theoretic universe but still use a countable pca such as $\mathbf{Kl}$ for the realisers. In this thesis we call such constructions *uninterpreted realisability models*, where we make the following modifications to the standard argument in section 2.3:

- (i) The domain U is a collection (i.e. set or definable class) in the base theory but not necessarily a subset of $\mathcal{A}$. For each term with parameters $t \in \mathcal{L}$, its interpretation $t^{\mathcal{A}}$ should be a function that maps assignments of its free variables to an element in U , but it need not be computable by $\mathcal{A}$.
- (ii) The realisability conditions for quantifiers are instead:

$$\begin{aligned} d \Vdash \forall x \varphi & \quad :\Longleftrightarrow \quad \forall a \in U \, d \Vdash \varphi[a/x], \\ d \Vdash \exists x \varphi & \quad :\Longleftrightarrow \quad \exists a \in U \, d \Vdash \varphi[a/x]. \end{aligned}$$

- (iii) For any formula (with parameters) $\varphi \in \mathcal{L}$ with its free variables enumerated as $x_1, \dots, x_k$, we still say that $d \in \mathcal{A}$ realises φ if and only if it realises the universal closure $\forall x_1, \dots, x_k \varphi$. But now this means that a uniform realiser d must satisfy

$$d \Vdash \varphi[a_1/x_1, \dots, a_k/x_k]$$

for any $\vec{a} = \langle a_1, \dots, a_k \rangle \in U^k$.

Proposition 2.4.1. *Soundness of intuitionistic logic as in Proposition 2.3.1 still holds for any uninterpreted realisability model.*

Proof. It suffices to check the axioms and inference rules involving quantifiers. Observe that:

- (11) $\lambda d. d$ simply realises $\forall x \varphi \rightarrow \varphi[t/x]$;
- (12) $\lambda d. d$ also realises $\varphi[t/x] \rightarrow \exists x \varphi$, since let $\vec{y}$ enumerate the free variables in this axiom, then for any corresponding tuple $\vec{a} \in U^k$, if d realises $\varphi[t^{\mathcal{A}}(\vec{a})/x, \vec{a}/\vec{y}]$, then it also realises $\exists x \varphi[\vec{a}/\vec{y}]$ by definition.

For the inference rules:

- (ii) Assume that $f \Vdash \varphi \rightarrow \psi$ where the free variables are enumerated as $x, \vec{y}$ and x does not occur free in φ , then for any corresponding $\vec{b} \in U^k$, if $d \Vdash \varphi[\vec{b}/\vec{y}]$, then $fd \Vdash \psi[a/x, \vec{b}/\vec{y}]$ for any $a \in U$. Thus $\lambda d. fd$ realises $\varphi \rightarrow \forall x \psi$;

- (iii) Assume that $f \Vdash \varphi \rightarrow \psi$ where the free variables are enumerated as $x, \vec{y}$ and x does not occur free in ψ , then for any corresponding $\vec{b} \in U^k$, if $d \Vdash \exists x \varphi[\vec{b}/\vec{y}]$, i.e. $d \Vdash \varphi[a/x, \vec{b}/\vec{y}]$ for some $a \in U$, then $fd \Vdash \psi[a/x, \vec{b}/\vec{y}]$ which is the same formula as $\psi[\vec{b}/\vec{y}]$. Thus $\lambda d. fd$ realises $\exists x \varphi \rightarrow \psi$. $\square$

2.5 Weak pcas

Finally, it is necessary to bring into this overview a notion of *weak pcas*, which consist of the same data as a pca, but with a relaxed requirement for the behaviour of the operation $\mathbf{s}$:

- (ii') for any $a, b, c \in \mathcal{A}$, $\mathbf{s}a \downarrow$, $\mathbf{s}ab \downarrow$, and

$$ac(bc) \downarrow \rightarrow \mathbf{s}abc \downarrow = ac(bc).$$

Thus, $\mathbf{s}abc$ may still converge when $ac(bc)$ diverges. This is a notion examined in [12, section 5.5], and can be seen as an intermediate variant between a pca and an ‘ordered partial combinatory algebra’ as considered by Hofstra and van Oosten [19]. It is pointed out in [13, section 5.4] that every weak pca can be made into a pca as defined in section 2.1 by manually introducing a correct $\mathbf{s}$ -combinator. However, as we will soon see later in section 3.2, we sometimes want to make use of non-conventional constructions that have an explicit definition for the application function, where it can be hard to require the element $\mathbf{s}$ to satisfy the strict rules of a pca. Thus, one may prefer to work with a weak variant directly, and we shall observe that many general observations in previous sections can be easily modified to accommodate a weak pca:

Lemma 2.5.1 (λ -abstraction). *Let $\mathcal{A}$ be a weak pca. For any formal application term t over variables $x, \vec{y}$, there is a formal application term (denoted as $\lambda x. t$) over variables $\vec{y}$ only, such that*

- (i) for any $\vec{b} \in \mathcal{A}^k$, $(\lambda x. t)[\vec{b}/\vec{y}] \downarrow$;

- (ii) for any $a \in \mathcal{A}$, $\vec{b} \in \mathcal{A}^k$,

$$t[a/x, \vec{b}/\vec{y}] \downarrow \rightarrow (\lambda x. t)[\vec{b}/\vec{y}]a \downarrow = t[a/x, \vec{b}/\vec{y}].$$

Proof. Same as Lemma 2.1.1. $\square$

Corollary 2.5.2 (Recursion theorem). *Let $\mathcal{A}$ be a weak pca. There must be a*

distinguished element $\mathbf{r} \in \mathcal{A}$ such that for any $f, a \in \mathcal{A}$, $\mathbf{r}f \downarrow$ and

$$fa(\mathbf{r}f) \downarrow \rightarrow \mathbf{r}fa \downarrow = fa(\mathbf{r}f).$$

Proof. Same as Corollary 2.1.2. □

Over the natural numbers, we mostly work with total functions, so nothing needs to be modified in order to define a *weak pca over natural numbers*. It is also easy to observe that realisability models can be built and Proposition 2.3.1 continues to hold for a weak pca over natural numbers.

Chapter 3

A semi-intuitionistic higher-order arithmetic

In this chapter, we work in the base theory of classical second-order arithmetic as developed in Simpson [43]. We build a realisability model for a theory of semi-intuitionistic third-order arithmetic proposed by Nik Weaver and determine the weakest fragment of second-order arithmetic required to formulate this interpretation, in order to perform an ordinal analysis of Weaver’s system. From section 3.4 onwards, we expand this classical mechanism to additionally analyse extensions of the system concerning a global well-ordering of the second-order objects.

The majority of content in this chapter is extracted from my arXiv manuscript [48] on Weaver’s system CM.

3.1 Weaver’s conceptual mathematics

Nik Weaver, who was a mathematician in the field of functional analysis, has published actively on arXiv about his ideas of a weak, essentially predicative foundation for mainstream mathematics since the 2000s. In this chapter, we focus on the formal fragment of third-order arithmetic he introduced in [53], as an example of his philosophical vision known as *mathematical conceptualism* [52]. In that exposition, he rejects the full Cantorian universe of sets as a foundational ontological commitment and instead advocates for a restricted language involving only countably and predicatively constructed mathematical objects; as he wrote in [52, section 1],

Classical set theory presents us with a picture of an incredibly vast

universe. [...] Yet virtually all important objects in mainstream mathematics are either countable or separable. This is not because the uncountable/nonseparable case has not yet been sufficiently studied but rather because on mainstream questions it tends to be either pathological or undecidable.

On the other hand, unlike the traditional Weylian predicativists who take the natural numbers as given and allow essentially only finitary constructions on top of them⁸, Weaver makes a hard distinction between the countable and uncountable collections. He believes that one is capable of mentally conceiving the completion of any countable process, for example through the well-known mechanism of supertasks in philosophy (see [54, section 3.1]); whereas uncountable processes, such as a transfinite enumeration of all real numbers, are inconceivable and thus do not deserve to be ‘complete’ objects in the mathematical ontology.

For formal conceptual mathematics, Weaver adopts a constructive notion of truth⁹ that aligns with his stance on ontology: he views a typical instance of classical reasoning

$$\forall x \in X \varphi \vee \exists x \in X \neg \varphi$$

as the statement that either the property φ holds for all $x \in X$, or that there is a *process* for finding one counterexample. Therefore, under a conceptualist philosophy, this is always possible when the domain X is countable, but very likely not the case when X is uncountable. This justifies the use of semi-intuitionistic logic in conceptual mathematics, i.e. intuitionistic logic together with the law of excluded middle $\varphi \vee \neg \varphi$ for formulae that only quantify over natural number variables, since it only requires enumerating over countably many objects to determine the truth value of such statements.

Now, we shall not be concerned with more philosophical justifications in this thesis, but grant that Weaver’s philosophical ideals explained so far motivate the following axioms in [53]:

- (1) the number axioms, essentially the axioms of HA excluding induction;

⁸This is most evident from the well-known identification of the ‘limit of predicativity’ as the proof-theoretic ordinal Γ_0 by Feferman [14] and Schütte [41], versus concerns raised by Howard [20] and Weaver [56, 57]. See page 58 for a brief discussion.

⁹Weaver has a flexible attitude regarding the underlying logic of mathematical reasoning and sees both classical formulations (such as the mini-fragment of set theory he worked with in [51]) and constructive ones (such as the system of conceptual mathematics in [53], which we study in this chapter) as justified under conceptualism. More detail is given in [54, section 2.4].

- (2) for any third-order formula φ , the full induction scheme:

$$\varphi(0) \wedge \forall n (\varphi(n) \rightarrow \varphi(n+1)) \rightarrow \forall n \varphi(n);$$

- (3) and the full recursion scheme (also known as *dependent choice*):

$$\forall n \forall X \exists Y \varphi(n, X, Y) \rightarrow \forall X \exists Z ((Z)_0 = X \wedge \forall n \varphi(n, (Z)_n, (Z)_{n+1}))$$

(where Z is a conventional encoding of a family of second-order objects $(Z)_n$ as a single object, to be explained in more detail on page 24), because both induction and recursion on first-order objects concern mere iterations of countable length;

- (4) for any third-order formula φ (which does not contain the new variable X or $\mathbf{X}$ used for comprehension), the *decidable comprehension* axioms:

$$\begin{aligned} \forall n (\varphi(n) \vee \neg \varphi(n)) &\rightarrow \exists X \forall n (n \in X \leftrightarrow \varphi(n)), \\ \forall X (\varphi(X) \vee \neg \varphi(X)) &\rightarrow \exists \mathbf{X} \forall X (X \in \mathbf{X} \leftrightarrow \varphi(X)), \end{aligned}$$

where X denotes a second-order variable and boldface $\mathbf{X}$ denotes a third-order variable;

- (5) and lastly, the law of excluded middle $\varphi \vee \neg \varphi$ for every arithmetic formula φ , i.e. formulae that do not contain second- or third-order quantifiers.

As Weaver demonstrated in [53, section 3], this formal theory is very expressive and suffices for the development of much mainstream mathematics such as real analysis, topology and functional analysis, by expressing arbitrary sets (e.g. $\mathbb{R}$, arbitrary topological spaces and arbitrary metric spaces) as third-order objects and their elements as second-order objects. For example, he defined and proved the following (separable) version of the Hahn–Banach theorem:

Theorem 3.1.1 ([53, Theorem 3.72], CM). *Let $\mathbf{E}$ be a separable Banach space, let $\mathbf{E}_0$ be a separable closed subspace, and let $\mathbf{f}_0 : \mathbf{E}_0 \rightarrow \mathbb{R}$ be a bounded linear functional on $\mathbf{E}_0$. Then $\mathbf{f}_0$ extends to a bounded linear functional $\mathbf{f}$ on $\mathbf{E}$ with $\|\mathbf{f}\| = \|\mathbf{f}_0\|$.*

The separability assumptions are needed because the restricted law of excluded middle and the decidable comprehension axioms in Weaver’s system together yield a deep distinction between quantifiers over countable and uncountable collections. Weaver was able to prove in his theory strong, classical results about algebraic structures that admit ‘countable descriptions’ such as a

countable dense set, which fail to apply to other structures in general. This is not a disheartening observation, as Weaver argued in [55], since the majority of mainstream functional analysis, for example, is precisely about separable spaces.

Thus, it is a very interesting question to ask what the precise proof-theoretic strength of Weaver's system is. In [37, section 2.7], Rathjen observed that this system is easily interpretable in the weak intuitionistic set theory $\text{CZF} + \text{LPO} + \text{RDC}$, which is shown in [34] to have the same proof-theoretic ordinal as CZF itself, namely the Bachmann–Howard ordinal. In this chapter, we will improve this bound and show that Weaver's theory is actually equiconsistent with the classical second-order theory of $\Sigma_1^1\text{-DC}$, which is much weaker and has proof-theoretic ordinal $\varphi_{\varepsilon_0}(0)$.

The full system CM

Before our analysis, we will first make a few clarifications and minor modifications to Weaver's original presentation of conceptual mathematics above, which do not affect the core ideas of his system.

We phrase *conceptual mathematics*, CM, as a theory in the usual three-sorted implementation of third-order logic, where first-order variables are denoted by lowercase letters $n, m, \dots$, second-order variables are denoted by italic uppercase letters $X, Y, \dots$, and third-order variables are denoted by bold uppercase letters $\mathbf{X}, \mathbf{Y}, \dots$. The vocabulary is that of first-order arithmetic, i.e. with the constant symbol 0 and function symbols for successor, $+$ and $\cdot$, as well as binary membership relations between first- and second-order and between second- and third-order objects, which will be distinguished and rendered¹⁰ as $n \in_1 X$ and $X \in_2 \mathbf{X}$.

Later in section 3.3, we will consider an extension of this theory envisioned by Weaver in [53, section 2.3], where a global well-ordering is hypothesised on the collection of all second-order objects. So we will preemptively introduce an additional binary relation symbol $\prec$ between two second-order variables. The only axiom in CM concerning it will be to assert that this is a decidable relation (as well as that second-order objects are extensional with respect to all relations, see axiom (8) later):

¹⁰This distinction is not made in Weaver's original formulation in [53]. However, we need this treatment so that, when we work with a realisability model and have the same kind of realisers (i.e. objects in the pca) assigned to all three sorts of variables, we can still identify the order of the variable from the presentation of the formula directly.

$$(6) \quad \forall X \forall Y (X \prec Y \vee \neg X \prec Y),$$

which is clearly conservative over the system without this symbol, for example, by trivially interpreting $X \prec Y$ as the decidable arithmetic formula $\forall n (n \in_1 X \leftrightarrow n \in_1 Y)$.

From [53, section 2.2], one can infer that Weaver would like to treat first-order equality as a primitive symbol with the standard equality axioms, while second- and third-order equalities are treated as extensional abbreviations:

$$\begin{aligned} X_1 = X_2 & \quad :\Longleftrightarrow \quad \forall n (n \in_1 X_1 \leftrightarrow n \in_1 X_2), \\ \mathbf{X}_1 = \mathbf{X}_2 & \quad :\Longleftrightarrow \quad \forall X (X \in_2 \mathbf{X}_1 \leftrightarrow X \in_2 \mathbf{X}_2). \end{aligned}$$

However, he does not include an axiom of extensionality for these inferred equalities in his formulation, which can bring quite some problems when we reason with third-order objects. For example, to understand expressions involving set-building terms like $\{n : \varphi(n)\} \in_2 \mathbf{X}$ as unambiguous abbreviations for formal formulae in the language — where the formula above is usually read as

$$\exists X (\forall n (n \in_1 X \leftrightarrow \varphi(n)) \wedge X \in_2 \mathbf{X})$$

— it is important that there are no multiple copies of the ‘same’ set of natural numbers with different behaviours in the theory. Therefore, we would add to CM the following axioms:

$$(7) \quad X = Y \rightarrow (X \in_2 \mathbf{X} \leftrightarrow Y \in_2 \mathbf{X}),$$

$$(8) \quad X = Y \wedge X' = Y' \rightarrow (X \prec Y \leftrightarrow X' \prec Y').$$

With these necessary ingredients, we can show that:

Lemma 3.1.2 (CM). *Given axioms (7) and (8), the Leibniz law holds, i.e.*

$$t = s \rightarrow (\varphi(t) \leftrightarrow \varphi(s)),$$

for any (third-order) formula φ and any terms t, s of the same, correct order.

Proof. This follows from simple induction on the complexity of φ . All atomic cases hold trivially by definition of the equality symbol, except when $\varphi(t)$ is $t \in_2 \mathbf{X}$, $t \prec X$ or $X \prec t$, where the extensionality assertions above are used. $\square$

Additionally, the arithmetic decidability scheme (5) can be complicated to

work with, and we observe that it can be broken down into two separate components:

(5') atomic decidability axioms:

$$n = m \vee \neg n = m, \quad n \in_1 X \vee \neg n \in_1 X, \quad X \in_2 \mathbf{X} \vee \neg X \in_2 \mathbf{X};$$

(5'') for any third-order formulae $\varphi(n)$ and $\psi(n)$, the numerical omniscience scheme, also known as the *limited principle of omniscience*:

$$\forall n (\varphi(n) \vee \psi(n)) \rightarrow \forall n \varphi(n) \vee \exists n \psi(n).$$

Lemma 3.1.3. *Over the rest of the axioms of CM, the axiom scheme (5) is equivalent to (5') + (5'').*

Proof. The forward direction is essentially [53, Theorem 2.2]. We only need to check numerical omniscience (5''): for any n , assume that $\varphi(n) \vee \psi(n)$, then clearly there exists some X such that

$$(\varphi(n) \wedge X = \emptyset) \vee (\psi(n) \wedge X = \{n\}).$$

Now, by the recursion axiom, there exists some Z such that $(Z)_n$ satisfies the condition for X above for each specific n (see the paragraphs after this proof for the formal definition of $(Z)_n$ using a recursive pairing function), and we can take $Y = \{n : n \in_1 (Z)_n\}$ by decidable comprehension, so that

$$\forall n ((\varphi(n) \wedge \neg n \in_1 Y) \vee (\psi(n) \wedge n \in_1 Y)).$$

By arithmetic decidability (5), we have $\neg \exists n n \in_1 Y \vee \exists n n \in_1 Y$ which immediately implies $\forall n \neg n \in_1 Y \vee \exists n n \in_1 Y$. Thus also $\forall n \varphi(n) \vee \exists n \psi(n)$ as desired.

For the backward direction, observe that (5'') implies

$$\begin{aligned} \forall n (\varphi(n) \vee \neg \varphi(n)) &\rightarrow \forall n \varphi(n) \vee \neg \forall n \varphi(n), \\ \forall n (\neg \psi(n) \vee \psi(n)) &\rightarrow \neg \exists n \psi(n) \vee \exists n \psi(n), \end{aligned}$$

i.e. the collection of formulae satisfying excluded middle is closed under first-order quantification. Since this collection is trivially also closed under logical connectives, arithmetic decidability (5) follows from atomic decidability (5')

through a straightforward induction on formula complexity. $\square$

In this thesis, we will take CM to formally denote the collection of axioms (1) + (2) + (3) + (4) + (5') + (5'') + (6) + (7) + (8).

Lastly, we shall use the notation $\langle -, - \rangle : \mathbb{N}^2 \rightarrow \mathbb{N}$ in CM as an abbreviation to denote the recursive bijection

$$\langle n, m \rangle = (n + m) \cdot (n + m + 1) / 2 + n,$$

with recursive inverses¹¹ $(-)_0, (-)_1 : \mathbb{N} \rightarrow \mathbb{N}$. We then follow [53, Definition 3.2] and specify the encoding for families of second-order objects used in the recursion scheme (3) as

$$(Z)_n = \{m : \langle n, m \rangle \in Z\}.$$

The fact that every natural number is a code for some pair will be convenient for constructions later. In [53], Weaver used a different, non-surjective map $\langle n, m \rangle = 2^n \cdot 3^m$. This difference does affect the precise meaning of the recursion scheme (3); however, since CM contains the classical subtheory PA, it can be easily shown that the different interpretations are ultimately equivalent.

3.2 A realisability model for CM

We begin the construction of our realisability model for CM in the classical base theory ACA_0 , which is defined and studied in detail in [43, Chapter III]. Following [43, Definition VIII.3.18], let $\pi(e, m, a, b, X)$ denote a universal Π_1^0 -formula with all free variables indicated, and correspondingly a universal Σ_1^1 -formula is given by $\exists X \pi(e, m, a, b, X)$, i.e. for any Σ_1^1 -formula $\varphi(m, a, b)$ with all free variables indicated, ACA_0 proves (cf. [43, Lemma V.1.4]) that there exists $e \in \mathbb{N}$ such that

$$\forall m \forall a \forall b (\varphi(m, a, b) \leftrightarrow \exists X \pi(e, m, a, b, X)).$$

We now let¹² Hyp denote the domain $\mathbb{N}$ with a partial application function

¹¹In this section, for any $a \in \mathbb{N}$, we use a_0, a_1 to denote the two components when a is treated as a pair of natural numbers. This is admittedly an abuse of notation, as we have previously defined $\mathbf{p}_0, \mathbf{p}_1$ to denote the unpairing functions in a pca. We shall clarify the ambiguity here by promising that we will never use $\mathbf{p}_0, \mathbf{p}_1$ to denote ‘the components of the natural number $\mathbf{p}$ ’ in this thesis.

¹²We pick the name Hyp from the fact that the total functions $\mathbb{N} \rightarrow \mathbb{N}$ representable in

$\text{ap} : \mathbb{N}^2 \rightarrow \mathbb{N}$ where we say $ea \downarrow = b$ if and only if $b \in \mathbb{N}$ is the *unique* natural number such that

$$\exists X \pi(e_0, e_1, a, b, X)$$

holds, where $e = \langle e_0, e_1 \rangle$. We will check below that it is a weak pca. To begin, we observe that:

Lemma 3.2.1. *Given any Σ_1^1 -formula $\varphi(m, a, b)$ with all free variables indicated, then there exists $e \in \mathbb{N}$ such that for any $m, a \in \mathbb{N}$, $em \downarrow$ always holds, and $ema \downarrow = b$ if and only if $b \in \mathbb{N}$ is the unique natural number such that $\varphi(m, a, b)$ holds.*

Proof. Simply pick $f, n \in \mathbb{N}$ by universality such that

$$\begin{aligned} \forall m \forall a \forall b (b = \langle m, a \rangle &\leftrightarrow \exists X \pi(f, m, a, b, X)), \\ \forall m \forall a \forall b (\varphi(m, a, b) &\leftrightarrow \exists X \pi(n, m, a, b, X)). \end{aligned}$$

Let $e = \langle f, n \rangle$, then $em \downarrow = \langle n, m \rangle$ for any $m \in \mathbb{N}$ and thus we have $ema \downarrow = b$ precisely when $b \in \mathbb{N}$ is the unique natural number such that $\varphi(m, a, b)$ holds. $\square$

Using this we can verify that:

Corollary 3.2.2. *Hyp is a weak pca over natural numbers.*

Proof. We use Lemma 3.2.1 to identify the distinguished elements in the algebra. For operations that involve one parameter only, we simply ignore the parameter m in the proposition:

- (i) let $s \in \mathbb{N}$ be the index such that for any $m, a \in \mathbb{N}$, $sma \downarrow = b$ if and only if $b = a + 1$, and set $\mathbf{s}_{\mathbb{N}} = s0$;
- (ii) let $p \in \mathbb{N}$ be the index such that for any $m, a \in \mathbb{N}$, $pma \downarrow = b$ if and only if $a = b + 1$, and set $\mathbf{p}_{\mathbb{N}} = p0$;

For $\mathbf{k}$ which involves two parameters, we use precisely the parameters in Lemma 3.2.1:

- (iii) let $\mathbf{k} \in \mathbb{N}$ simply be the index such that for any $a, b \in \mathbb{N}$, $\mathbf{k}ab \downarrow = c$ if and only if $c = a$.

the weak pca are precisely the hyperarithmetical functions. However, this cannot be verified in a weak base theory like ACA_0 ; we will later see as Theorem 3.4.5 that this is provable in ATR_0 (cf. [43, Theorem VIII.3.19]).

For $\mathbf{s}$ which involves three parameters, we invoke the proposition twice:

- (iv) let $s' \in \mathbb{N}$ be the index such that for any $a, b \in \mathbb{N}$, $s'ab \downarrow = c$ if and only if $c \in \mathbb{N}$ is the unique natural number satisfying¹³

$$\exists X, Y, Z \exists d, e (\pi(a_{00}, a_{01}, b, d, X) \wedge \pi(a_{10}, a_{11}, b, e, Y) \wedge \pi(d_0, d_1, e, c, Z)).$$

Thus, if $a_0b \downarrow = d$, $a_1b \downarrow = e$ and $de \downarrow = c$, then also $s'ab \downarrow = c$. Now let $\mathbf{s} \in \mathbb{N}$ be such that $\mathbf{s}ab \downarrow = s' \langle a, b \rangle$, i.e. $\mathbf{s}ab \downarrow = c$ if and only if

$$\exists X \pi(s'_0, s'_1, \langle a, b \rangle, c, X).$$

Finally, similarly for $\mathbf{d}_{\mathbb{N}}$ which involves four parameters:

- (v) let $d' \in \mathbb{N}$ be the index such that for any $a, b \in \mathbb{N}$,

$$d'ab \downarrow = \begin{cases} a_1 & \text{if } a_{00} = a_{01}, \\ b & \text{if } a_{00} \neq a_{01}. \end{cases}$$

Then let $d'' \in \mathbb{N}$ be such that $d''ab \downarrow = d' \langle a, b \rangle$ and finally $\mathbf{d}_{\mathbb{N}} \in \mathbb{N}$ be such that $\mathbf{d}_{\mathbb{N}}ab \downarrow = d'' \langle a, b \rangle$. $\square$

We will stipulate that the standard pairing function denoted by $\mathbf{p}$ in Hyp is

$$\mathbf{p}ab = \langle a, b \rangle,$$

which can be computed in exactly the same way as in the proof above. In pca contexts we will just write $\mathbf{p}nm$ for $\langle n, m \rangle$, $\mathbf{p}_0n$ for n_0 and $\mathbf{p}_1n$ for n_1 .

Now, consider a third-order realisability model over the weak pca Hyp, where variables of all three sorts are interpreted using domains within the weak pca. A first-order variable shall range over $\mathbb{N}$, a second-order variable shall range over the definable class

$$S_2 = \{d \in \text{Hyp} : \forall n \, dn \downarrow \in \{0, 1\}\},$$

and third-order variables range over

$$S_3 = \{d \in \text{Hyp} : \forall f \in S_2 \, df \downarrow \in \{0, 1\} \wedge \forall e, f \in S_2 (e \equiv f \rightarrow de = df)\},$$

¹³Note that the existence of a unique c here does not also guarantee the uniqueness of the intermediate values d and e . Thus the existence of a unique c does not imply $a_0b(a_1b) \downarrow$ and Hyp as constructed here is only a weak pca.

where the binary relation $\equiv$ of extensional equivalence on S_2 is defined as

$$e \equiv f \quad :\Longleftrightarrow \quad \forall n \, en = fn.$$

We fix an arbitrary interpretation $q \in \text{Hyp}$ for the relation $\prec$ in the language of CM satisfying $qee'\downarrow \in \{0, 1\}$ for any $e, e' \in S_2$, and

$$e \equiv f \wedge e' \equiv f' \rightarrow qee' = qff'$$

for any $e, e', f, f' \in S_2$. The precise choice of q does not matter for interpreting the base theory CM, so we can just take, for example, $q = \lambda nm.0$.

The realisability conditions are then given as in section 2.3, with the atomic clauses given by

$$\begin{aligned} d \Vdash t = s & \quad :\Longleftrightarrow \quad t = s \text{ for any arithmetic terms } t \text{ and } s, \\ d \Vdash t \in_1 e & \quad :\Longleftrightarrow \quad \exists n (t = n \wedge en = 1) \text{ for any arithmetic term } t, \\ d \Vdash e \in_2 f & \quad :\Longleftrightarrow \quad fe = 1, \\ d \Vdash e \prec f & \quad :\Longleftrightarrow \quad qef = 1. \end{aligned}$$

We now move into the extension of ACA_0 known as $\Sigma_1^1\text{-AC}$ (with unrestricted induction), which adds the axiom of second-order choice

$$\forall n \, \exists Y \, \varphi(n, Y) \rightarrow \exists Z \, \forall n \, \varphi(n, (Z)_n)$$

for any Σ_1^1 -formula φ , and induction on the natural numbers for any second-order formulae. Here we show that the axioms of CM are all realised.

Firstly, the equality axioms for the primitive, first-order equality symbol are realised trivially here. The number axioms (1) are just true, quantifier-free formulae in the base theory (with free variables), which can be easily realised by realisers that are total on the appropriate inputs.

For schemes (2) and (3), we use the unrestricted induction axiom in $\Sigma_1^1\text{-AC}$ with recursion in the weak pca :

Lemma 3.2.3. *The induction scheme (2) is realised:*

$$\varphi(0) \wedge \forall n (\varphi(n) \rightarrow \varphi(n+1)) \rightarrow \forall n \, \varphi(n).$$

Proof. Let $n, \vec{x}$ enumerate the free variables in φ , where $\vec{x}$ can include variables

of any order. If $d \Vdash \varphi[0/n, \vec{a}/\vec{x}]$ and $f \Vdash \forall n (\varphi(n) \rightarrow \varphi(n+1)) [\vec{a}/\vec{x}]$, then

$$\mathbf{r}(\lambda yt. \mathbf{d}_{\mathbb{N}}^* y 0 d(f(\mathbf{p}_{\mathbb{N}} y)(t(\mathbf{p}_{\mathbb{N}} y)))) m \Vdash \varphi[m/n, \vec{a}/\vec{x}]$$

by induction¹⁴ on $m \in \mathbb{N}$. Thus $\lambda \vec{x} d f. \mathbf{r}(\lambda yt. \mathbf{d}_{\mathbb{N}}^* y 0 d(f(\mathbf{p}_{\mathbb{N}} y)(t(\mathbf{p}_{\mathbb{N}} y))))$ realises the induction axiom as desired. $\square$

Lemma 3.2.4. *The recursion scheme (3) is realised:*

$$\forall n \forall X \exists Y \varphi(n, X, Y) \rightarrow \forall X \exists Z ((Z)_0 = X \wedge \forall n \varphi(n, (Z)_n, (Z)_{n+1}))$$

Proof. Let $n, X, Y, \vec{x}$ enumerate the free variables in φ , where $\vec{x}$ can include variables of any order. Given $f \Vdash \forall n \forall X \exists Y \varphi(n, X, Y) [\vec{a}/\vec{x}]$ and $s \in S_2$, we compute

$$g = \mathbf{r}(\lambda yt. \mathbf{d}_{\mathbb{N}}^* y 0 s(\mathbf{p}_0(f(\mathbf{p}_{\mathbb{N}} y)(t(\mathbf{p}_{\mathbb{N}} y))))),$$

then $g0 = s$ and $g(\mathbf{s}_{\mathbb{N}} n)$ is the ‘next’ set Y after some X represented by gn , as given by f . Thus, the set Z can be computed as

$$z = \lambda m. g(\mathbf{p}_0 m)(\mathbf{p}_1 m) \in S_2.$$

Now $(Z)_0 = X$ can be interpreted as

$$\forall n \exists y (y = \langle 0, n \rangle \wedge (y \in_1 Z \leftrightarrow n \in_1 X))$$

and realised simply by $\lambda n. \mathbf{p}(\mathbf{p}_0 n)(\mathbf{p}_0(\mathbf{p}(\lambda d. 0)(\lambda d. 0)))$, whereas $\mathbf{p}_1(fn(gn))$ realises¹⁵ $\varphi(n, (Z)_n, (Z)_{n+1})$ by induction on $n \in \mathbb{N}$. $\square$

The decidable comprehension axioms (4) are easy cases here:

¹⁴Even though we never use an arbitrary amount of second-order quantifiers, the use of unrestricted induction is necessary here: suppose that φ contains an alternating sequence of quantifiers, then the statement $d \Vdash \varphi(p)$ expands to a form like

$$\forall n_0 \exists d_1 (dn_0 \downarrow = d_1 \wedge \forall n_1 \exists d_2 ((\mathbf{p}_0 d_1)n_1 \downarrow = d_2 \wedge \dots \forall n_i \exists d_i \dots$$

Here formulae of the form $dn \downarrow = e$ are Δ_2^1 but lie within a sequence of first-order quantifiers. Since Σ_1^1 -AC does not show that the class of Δ_2^1 -formulae is closed under adding first-order quantifiers, it cannot prove that this statement has bounded complexity.

¹⁵To be completely formal, we will need to understand $\varphi(n, (Z)_n, (Z)_{n+1})$ as

$$\exists X \exists Y ((Z)_n = X \wedge (Z)_{n+1} = Y \wedge \varphi(n, X, Y)),$$

so the full realiser would be $\mathbf{p}(gn)(\mathbf{p}(g(\mathbf{s}_{\mathbb{N}} n))(\mathbf{p}d(\mathbf{p}_1(fn(gn)))))$ where d is the uninteresting realiser for $(Z)_n = X \wedge (Z)_{n+1} = Y$.

Lemma 3.2.5. *The following comprehension axioms are realised:*

$$\begin{aligned} \forall n (\varphi(n) \vee \neg \varphi(n)) &\rightarrow \exists X \forall n (n \in X \leftrightarrow \varphi(n)), \\ \forall X (\varphi(X) \vee \neg \varphi(X)) &\rightarrow \exists \mathbf{X} \forall X (X \in \mathbf{X} \leftrightarrow \varphi(X)), \end{aligned}$$

Proof. Let $n, \vec{x}$ enumerate the free variables in $\varphi(n)$. Given any realiser $f \Vdash \forall n (\varphi(n) \vee \neg \varphi(n))[\vec{a}/\vec{x}]$, the set X we need is simply

$$s = \lambda n. 1 - \mathbf{p}_0(fn) \in S_2,$$

and we have $\lambda d. \mathbf{p}_1(fm) \Vdash (n \in X \rightarrow \varphi(n))[m/n, s/X, \vec{a}/\vec{x}]$. Thus we know that $\lambda \vec{x} f. \mathbf{p}(\lambda n. 1 - \mathbf{p}_0(fn))(\lambda n. \mathbf{p}(\lambda d. \mathbf{p}_1(fn))(\lambda d. 0))$ realises the (decidable) second-order comprehension axiom.

The third-order case is entirely analogous. □

For the atomic decidability axioms (5'), simply observe that:

- $t = s \vee \neg t = s$ is realised by

$$\lambda \vec{x}. \begin{cases} \mathbf{p}00 & \text{if } t = s, \\ \mathbf{p}10 & \text{if } t \neq s, \end{cases}$$

where $\vec{x}$ enumerate the free variables in the terms t and s and are used in the computation to distinguish the cases;

- $t \in_1 e \vee \neg t \in_1 e$ is realised by $\lambda \vec{x} e. \mathbf{p}(1 - et)0$ where $\vec{x}$ enumerate the free variables and are used to compute t ;
- $e \in_2 f \vee \neg e \in_2 f$ is realised by $\lambda e f. \mathbf{p}(1 - fe)0$.

And likewise the decidability of the binary relation (6), $e \prec f \vee \neg e \prec f$, is realised by $\lambda e f. \mathbf{p}(1 - qef)0$.

Lemma 3.2.6. *The numerical omniscience scheme (5'') is realised:*

$$\forall n (\varphi(n) \vee \psi(n)) \rightarrow \forall n \varphi(n) \vee \exists n \psi(n).$$

Proof. Let $\vec{x}$ enumerate the free variables in $\forall n (\varphi(n) \vee \psi(n))$ and suppose that $f \Vdash \forall n (\varphi(n) \vee \psi(n))[\vec{a}/\vec{x}]$. By Σ_1^1 -choice, it is then the case that

$$\exists Z \forall n \exists ! b \pi(f_0, f_1, n, b, (Z)_n)$$

where the number b corresponding to the same $n \in \mathbb{N}$ is necessarily unique.

By Lemma 3.2.1, there is $e \in \text{Hyp}$ such that for any $f \in \mathbb{N}$, $ef \downarrow = d$ if and only if

- (a) either $\exists Z \forall n \exists b (\pi(f_0, f_1, n, b, (Z)_n) \wedge b_0 = 0)$ and $d = \langle 0, 0 \rangle$,
- (b) or $\exists Z \exists n, b (\pi(f_0, f_1, n, b, (Z)_n) \wedge b_0 = 1 \wedge d = \langle 1, \langle n, b_1 \rangle \rangle)$, where n is the least such number, i.e. also $\forall k < n \forall b (\pi(f_0, f_1, k, b, (Z)_k) \rightarrow b_0 = 0)$.

It is then easy to verify that $f \Vdash \forall n (\varphi(n) \vee \psi(n))[\vec{a}/\vec{x}]$ implies $ef \downarrow$, where either $\forall n \mathbf{p}_0(fn) = 0$ and $ef = \langle 0, 0 \rangle$, or $ef = \langle 1, \langle m, d \rangle \rangle$ for some $m, d \in \mathbb{N}$ such that $fm = \langle 1, d \rangle$, i.e. $d \Vdash \psi[m/n, \vec{a}/\vec{x}]$.

Hence, $\lambda \vec{x} f. \mathbf{d}_{\mathbb{N}}(\mathbf{p}_0(ef))0(\mathbf{p}_0(\lambda y. \mathbf{p}_1(fy)))(\mathbf{p}_1(\mathbf{p}_1(ef)))$ realises the axiom as desired. $\square$

Finally, observe that in the set-up we have ensured for any $d \in S_3$,

$$\forall e, f \in S_2 (e \equiv f \rightarrow de = df),$$

and likewise for the fixed interpretation q of the relation $\prec$,

$$\forall e, e', f, f' \in S_2 (e \equiv f \wedge e' \equiv f' \rightarrow qee' = qff').$$

Thus the extensionality assertions (7) and (8) will both be realised by trivial realisers of the form $\lambda \vec{x} d. \mathbf{p}(\lambda e. 0)(\lambda e. 0)$.

Combining the lemmata above and Proposition 2.3.1, we can conclude that:

Theorem 3.2.7. *Let φ be any theorem of CM, then $\Sigma_1^1\text{-AC}$ proves there exists $d \in \text{Hyp}$ such that $d \Vdash \varphi$.*

An ordinal analysis of CM

Let $\Sigma_1^1\text{-DC}$ denote the extension of ACA_0 by unrestricted induction and the axiom of Σ_1^1 -dependent choice (cf. [43, section VII.6]), $\Delta_1^1\text{-CA}$ denote the extension of ACA_0 by unrestricted induction and the axiom of Δ_1^1 -comprehension (cf. [43, section VII.5]), and $\Sigma_1^1\text{-AC}^i$ denote the intuitionistic fragment of $\Sigma_1^1\text{-AC}$. It is well-known that:

Theorem 3.2.8. *The following theories have the same proof-theoretic strength:*

$$\Sigma_1^1\text{-AC}^i \equiv_{\text{Con}} \Sigma_1^1\text{-AC} \equiv_{\text{Con}} \Sigma_1^1\text{-DC}.$$

Proof. The proof-theoretic equivalence of $\Sigma_1^1\text{-AC}^i$ and $\Sigma_1^1\text{-AC}$ is due to Peter

Aczel [1]. Harvey Friedman showed in [15] that one can interpret both $\Sigma_1^1\text{-AC}$ and $\Sigma_1^1\text{-DC}$ in $\Delta_1^1\text{-CA}$, thus the three theories are equivalent in strength since $\Delta_1^1\text{-comprehension}$ is implied by $\Sigma_1^1\text{-choice}$. $\square$

Since $\Sigma_1^1\text{-AC}^i$ is an obvious subtheory of CM and our analysis above shows that CM can be interpreted in $\Sigma_1^1\text{-AC}$, it is an immediate corollary that:

Corollary 3.2.9. $\text{CM} \equiv_{\text{Con}} \Sigma_1^1\text{-DC}$. *In particular, since the exact proof-theoretic strength of $\Sigma_1^1\text{-DC}$ is analysed in [16], we have $|\text{CM}| = |\Sigma_1^1\text{-DC}| = \varphi_{\varepsilon_0}(0)$.*

3.3 Axiomatising a global well-ordering

In [53, section 2.3], Weaver suggested that the axioms of CM do not fully represent the conception that ‘second-order objects [appear] in well-ordered stages’ and briefly considered extending the system to include the axiomatisation of a global well-ordering on the collection of second-order objects. Formally, this extension would include the following axioms:

- (W1) the binary relation $\prec$ between second-order objects is a linear ordering;
- (W2) the transfinite induction scheme

$$\forall X (\forall Y (Y \prec X \rightarrow \varphi(Y)) \rightarrow \varphi(X)) \rightarrow \forall X \varphi(X),$$

for a certain class of formulae;

- (W3) the axiom of countable initial segments

$$\forall X \exists Z \forall Y (Y \prec X \rightarrow \exists n Y = (Z)_n).$$

For the axiom scheme (W2) of transfinite induction, Weaver made explicit that an extension justified in his conceptualist philosophy cannot assert this for every formula φ — especially those that contain the relation $\prec$ itself — since the collection of all second-order objects is uncountable and from a predicativist perspective, Weaver does not want to assert that they can be enumerated in a way that supports induction using properties of the entire well-ordering (cf. [56, section 2.5]).

However, in his brief discussion of the extension, Weaver never managed to suggest a plausible class of formulae to which the transfinite induction scheme

can be restricted. We will give a detailed analysis of his proposal in [53, section 2.3] and other possible alternatives of a ‘predicative’ global well-ordering later in section 3.6. For now, we shall let GWO denote the entire collection of axioms (W1) + (W2) + (W3) where the transfinite induction scheme (W2) applies to any third-order formula φ in the language. We will examine the expressive power of the full extension CM + GWO in this section and provide an ordinal analysis for it in section 3.4, which acts as an upper bound for the strength of all its (predicative) fragments.

Zorn’s lemma and applications

In [53, section 3], Weaver gave a very detailed exposition of how ordinary mathematics, including theorems of real analysis, topology and functional analysis, can be treated inside CM, in order to justify its viability as a foundational theory of mathematics. However, he never explored how the global well-ordering axioms add to this development — whether they fulfil the role of a ‘genuine choice axiom’ for mathematics as he suggested — apart from a comment after his proof of the separable Hahn–Banach theorem in CM (previously mentioned in this thesis as Theorem 3.1.1) that:

The same result holds, with the same proof, for extensions from separable subspaces of nonseparable spaces, but this requires the well-ordering $\prec$.

In this section, we will prove inside CM + GWO a general result analogous to Zorn’s lemma and demonstrate that the global well-ordering axioms indeed serve their purpose as a choice principle and constitute a useful extension if one wants to use CM as a foundational theory. This shall serve as a typical sample of how much transfinite induction is needed by mainstream mathematics and facilitate our later discussion of the (usable) predicative fragments of GWO in section 3.6.

To begin, we will first observe that omniscience holds for a bounded second-order quantifier:

Lemma 3.3.1 (CM + GWO). *For any formula $\varphi(X)$ and $\psi(X)$, we have*

$$\forall X (\forall Y \prec X (\varphi(Y) \vee \psi(Y)) \rightarrow (\forall Y \prec X \varphi(Y) \vee \exists Y \prec X \psi(Y))) .$$

Obviously the same also holds when every $\prec$ is replaced by $\preceq$ in the proposition.

Proof. The axiom of countable initial segments asserts that for any second-

order set X , there exists Z satisfying

$$\forall Y \prec X \exists n Y = (Z)_n.$$

Assume that $\forall Y \prec X (\varphi(Y) \vee \psi(Y))$. For each $n \in \mathbb{N}$, we know by atomic decidability that $(Z)_n \prec X \vee \neg(Z)_n \prec X$, and in the former case it follows that $\varphi((Z)_n) \vee \psi((Z)_n)$. By numerical omniscience, we have

$$\forall n ((Z)_n \prec X \rightarrow \varphi((Z)_n)) \vee \exists n ((Z)_n \prec X \wedge \psi((Z)_n)).$$

The former disjunct implies $\forall Y \prec X \varphi(Y)$, while the latter disjunct implies $\exists Y \prec X \psi(Y)$. $\square$

We also need a construction to encode countable collections of second-order objects as one single second-order object in CM. Instead of the obvious coding using slices $(Z)_n$, we will use a slightly modified construction so that we can accommodate the empty or finite collections more conveniently: for any second-order set X , let $[X]_n = \{m : \langle n, 0 \rangle \in_1 X \wedge \langle n, m+1 \rangle \in_1 X\}$, and define a new binary relation $\in^*$ between two second-order objects as

$$Y \in^* X \quad :\Longleftrightarrow \quad \exists n (\langle n, 0 \rangle \in_1 X \wedge Y = [X]_n).$$

It follows that we can denote the following simple abbreviations:

$$\begin{aligned} X \subseteq^* Y & \quad :\Longleftrightarrow \quad \forall Z \in^* X \ Z \in^* Y, \\ X \subseteq^* \mathbf{X} & \quad :\Longleftrightarrow \quad \forall Z \in^* X \ Z \in_2 \mathbf{X}, \\ X =^* Y & \quad :\Longleftrightarrow \quad X \subseteq^* Y \wedge Y \subseteq^* X, \end{aligned}$$

and immediately observe that omniscience for countable collections,

$$\forall Y \in^* X (\varphi(Y) \vee \psi(Y)) \rightarrow (\forall Y \in^* X \varphi(Y) \vee \exists Y \in^* X \psi(Y)),$$

follows directly from numerical omniscience, thus the relations defined here are all decidable, i.e. the law of excluded middle holds for them.

In the same way, we can also denote certain set-forming expressions as second-order terms, such as:

$$\begin{aligned} \{X_1, \dots, X_k\} &= \{\langle n, m \rangle : 1 \leq n \leq k \wedge (m = 0 \vee \exists i \in_1 X_n \ m = i + 1)\}, \\ \bigcup X &= \{\langle \langle n, m \rangle, i \rangle : \langle n, 0 \rangle \in_1 X \wedge \langle n, \langle m, i \rangle + 1 \rangle \in_1 X\}, \end{aligned}$$

$$\{Y \in^* X : \varphi(Y)\} = \{\langle n, m \rangle : \langle n, 0 \rangle \in_1 X \wedge \langle n, m \rangle \in_1 X \wedge \varphi([X]_n)\},$$

where in the last definition, the formula $\varphi(Y)$ must be decidable. It is easy to verify that these collections ‘contain’ the same second-order sets as their third-order counterparts.

Now, an analogue of Zorn’s lemma can be stated as follows¹⁶:

Proposition 3.3.2 (CM + GWO). *Let $\varphi(X)$ be a decidable formula (possibly with other parameters) where X denotes a countable second-order collection encoded as above, i.e. we have $\forall X (\varphi(X) \vee \neg\varphi(X))$. If φ is downward closed, i.e.*

$$\forall X, Y ((\varphi(X) \wedge Y \subseteq^* X) \rightarrow \varphi(Y)),$$

and also closed under unions of countable chains, i.e.

$$\forall X \left((\forall Y, Z \in^* X (Y \subseteq^* Z \vee Z \subseteq^* Y) \wedge \forall Y \in^* X \varphi(Y)) \rightarrow \varphi\left(\bigcup X\right) \right),$$

then there exists some $\mathbf{X}$ satisfying $\forall X \subseteq^ \mathbf{X} \varphi(X)$, which is maximal among such subsets in the sense that*

$$\forall Y (Y \notin \mathbf{X} \rightarrow \exists Z \subseteq^* \mathbf{X} \neg\varphi(Z \cup \{Y\})).$$

Lemma 3.3.3 (CM + GWO). *For any X , there exists Y such that*

$$\forall n (X)_n \prec Y.$$

The same immediately holds for the countable collection encoding we defined above, i.e. for any X , there exists Y such that $\forall Z \in^ X Z \prec Y$.*

Proof. By the axiom of countable initial segments,

$$\forall n \exists W \forall Y \preceq (X)_n \exists m Y = (W)_m,$$

and by the recursion axiom, we can find some U satisfying that

$$\forall n \forall Y \preceq (X)_n \exists m Y = ((U)_n)_m.$$

¹⁶The proof of Zorn’s lemma presented here is a slightly modified version of the proof in [48, Proposition 3.2]. The readers should observe that the presentation in this thesis only invokes the transfinite induction axiom for formulae with a single unbounded second-order existential quantifier (on top of existing quantifiers in φ), which will be important for discussions in section 3.6.

We shall diagonalise and construct

$$Y_0 = \{ \langle n, m \rangle : \langle n, m \rangle \notin_1 ((U)_n)_m \},$$

so that for any n , $\neg Y_0 \preceq (X)_n$. By linearity of the well-ordering, it must follow that $\forall n (X)_n \prec Y_0$.

For a countable family X encoded through $\in^*$ instead, simply apply the same process to

$$X' = \{ \langle n, m \rangle : \langle n, m+1 \rangle \in X \}.$$

Clearly $Z = [X]_n$ implies $\exists n Z = (X')_n$, and thus $\forall Z \in^* X Z \prec Y_0$ for Y_0 constructed as above. $\square$

Proof of Proposition 3.3.2. Define formulae

$$\begin{aligned} \psi(X, Z) &: \iff \varphi(Z) \wedge (\varphi(Z \cup \{X\}) \rightarrow X \in^* Z), \\ \theta(X, Z) &: \iff \forall Y \in^* Z Y \preceq X \wedge \forall Y \preceq X \psi(Y, Z^{\preceq Y}). \end{aligned}$$

where $Z^{\preceq Y} = \{W \in^* Z : W \preceq Y\}$. Fix any Z_1, Z_2 , and we first show by transfinite induction on X that

$$\forall X (\theta(X, Z_1^{\preceq X}) \wedge \theta(X, Z_2^{\preceq X}) \rightarrow Z_1^{\preceq X} =^* Z_2^{\preceq X}).$$

This is because for any $Y \prec X$, $\theta(X, Z^{\preceq X})$ clearly implies $\theta(Y, Z^{\preceq Y})$. Thus from the inductive hypothesis, we know that $\forall Y \prec X Z_1^{\preceq Y} =^* Z_2^{\preceq Y}$, and clearly $Z_1^{\preceq X} =^* \bigcup_{Y \prec X} Z_1^{\preceq Y} =^* \bigcup_{Y \prec X} Z_2^{\preceq Y} =^* Z_2^{\preceq X}$. Now assume that $X \in^* Z_1$, then

$$Z_1^{\preceq X} =^* Z_1^{\prec X} \cup \{X\} =^* Z_2^{\prec X} \cup \{X\} =^* Z_2^{\preceq X} \cup \{X\},$$

so $\varphi(Z_2^{\preceq X} \cup \{X\})$ holds and thus $X \in^* Z_2$. By symmetry, the converse can be shown assuming $X \in^* Z_2$, so we can conclude that $Z_1^{\preceq X} =^* Z_2^{\preceq X}$.

For any X, Z , observe that $\theta(X, Z)$ implies $Z =^* Z^{\preceq X}$. Thus the induction above implies that for any X , the second-order object Z satisfying $\theta(X, Z)$ is unique up to $=^*$, i.e.

$$\forall X, Z_1, Z_2 (\theta(X, Z_1) \wedge \theta(X, Z_2) \rightarrow Z_1 =^* Z_2).$$

We then show by another transfinite induction that $\forall X \exists Z \theta(X, Z)$. Fix X and pick W by the axiom of countable initial segments such that $\forall Y \prec X \exists n Y =$

$(W)_n$, and the inductive hypothesis would imply

$$\forall n \exists Z ((W)_n \prec X \rightarrow \theta((W)_n, Z)).$$

By the recursion axiom in CM, we find some second-order U satisfying that $\forall n ((W)_n \prec X \rightarrow \theta((W)_n, (U)_n))$, and now define

$$Z' = \{ \langle n, m \rangle : (W)_n \prec X \wedge (m = 0 \vee \exists i \in_1 (U)_n m = i + 1) \},$$

essentially the countable collection $\{(U)_n : (W)_n \prec X\}$. For each $Y \in^* Z'$, we then have $\varphi(Y)$. Z' is additionally a chain because for any $[Z']_m, [Z']_n \in^* Z'$, we can assume without loss of generality that $(W)_m \preceq (W)_n$, and thus

$$[Z']_m =^* [Z']_n^{\preceq (W)_m} \subseteq^* [Z']_n$$

by uniqueness above. Hence $\varphi(\bigcup Z')$. It also follows that $[Z']_n = (\bigcup Z')^{\preceq (W)_n}$ for any $n \in \mathbb{N}$ such that $(W)_n \prec X$.

Therefore, it suffices to set $Z = (\bigcup Z') \cup \{X\}$ if $\varphi((\bigcup Z') \cup \{X\})$ holds and $Z = \bigcup Z'$ otherwise (which is possible because φ is decidable). It is easy to verify that $\theta(X, Z)$ holds indeed.

Finally, by the unique existence of some Z satisfying $\theta(Y, Z)$ up to $=^*$, we can decidablely comprehend the diagonal set

$$\mathbf{X} = \{Y : \exists Z (\theta(Y, Z) \wedge Y \in^* Z)\}.$$

For any countable subset $X \subseteq^* \mathbf{X}$, by Lemma 3.3.3, there exists Y_0 such that $\forall Y \in^* X Y \prec Y_0$. Let Z_0 be a countable collection satisfying $\theta(Y_0, Z_0)$, then our definition of $\mathbf{X}$ and the uniqueness of Z_0 ensure that for any $Y \preceq Y_0$,

$$Y \in_2 \mathbf{X} \leftrightarrow Y \in^* Z_0.$$

In other words, $X \subseteq^* Z_0$, and $\varphi(X)$ follows immediately from $\varphi(Z_0)$ by downward closedness.

On the other hand, suppose that some $Y \notin_2 \mathbf{X}$. There must exist Z satisfying $\theta(Y, Z)$ and $Y \notin^* Z$, which, by our stipulation of ψ at the beginning of this proof, implies $\neg\varphi(Z \cup \{Y\})$. The uniqueness of Z ensures that also $Z \subseteq^* \mathbf{X}$. Therefore, we have checked that $\mathbf{X}$ has all the desired properties. $\square$

This proposition can be applied just like the classical Zorn's lemma to show

that one can find maximal subsets satisfying certain properties in uncountable collections, as long as the corresponding decidability criterion can be verified through intuitionistic reasoning. For example, consider any field $\mathbf{F}$ and an $\mathbf{F}$ -vector space $\mathbf{V}$, with their respective unary and binary operations also given as third-order objects, following the conventions of [53, section 3], then it follows that:

Corollary 3.3.4 (CM + GWO, [48, Corollary 3.3]). *Suppose that the formula*

$$\exists Y \left(\forall n \leq k (Y)_n \in \mathbf{F} \wedge \exists n \leq k (Y)_n \neq 0_{\mathbf{F}} \wedge 0_{\mathbf{V}} = \sum_{n=0}^k (Y)_n \cdot (X)_n \right),$$

i.e. informally the statement ‘ $(X)_0, \dots, (X)_k$ are linearly dependent as vectors in $\mathbf{V}$ ’, is decidable (whenever $(X)_0, \dots, (X)_k \in_2 \mathbf{V}$), then $\mathbf{V}$ has a basis $\mathbf{X} \subseteq \mathbf{V}$ such that every vector $V \in \mathbf{V}$ is spanned by finitely many elements in $\mathbf{X}$.

This applies to show the existence of a basis in the following scenarios:

- (a) when the field $\mathbf{F}$ is *countable* — for example, when showing that the Hamel basis for the reals $\mathbb{R}$ as a vector space over the rationals $\mathbb{Q}$ exists;
- (b) when the vector space $\mathbf{V}$ has an *inner product* $\mathbf{V}^2 \rightarrow \mathbf{F}$;
- (c) when $\mathbf{F}$ is a complete $\mathbb{R}$ -normed field with a countable dense set $C \subseteq^* \mathbf{F}$ with respect to the induced norm, and $\mathbf{V}$ has a compatible norm $\|-\| : \mathbf{V} \rightarrow \mathbf{R}$ — for example, when $\mathbf{V}$ is any real or complex Banach space.

The proofs of these results are sketched in [48], but are omitted in this thesis since they are not related to the topic of realisability models. At any rate, I think they demonstrate the relevance of the global well-ordering axioms to mathematics founded on the system CM and justify our endeavour to analyse the proof-theoretic commitments in them.

3.4 Realising the axioms of GWO

Now, we look back at the same realisability model for CM as in section 3.2, but in order to realise the axioms of GWO, we will need a stronger base theory in which we can actually construct a well-ordering on our domain for second-order variables,

$$S_2 = \{d \in \text{Hyp} : \forall n \, dn \downarrow \in \{0, 1\}\}.$$

We shall do this by developing the theory of hyperarithmetic sets in the theory ATR_0 , consisting of ACA_0 and the additional axiom of arithmetic transfinite

recursion (cf. [43, section V.2]), and show that the (total) functions denoted by indices in S_2 are precisely the hyperarithmetic functions.

Hyperarithmetic theory in ATR_0

The constructions we will need here roughly follow [43, section VIII.3]: let $\tilde{\pi}(e, m, X)$ again¹⁷ denote a universal Π_1^0 -formula with all free variables indicated, such that for any Π_1^0 -formula $\varphi(e', m, X)$ with the same number of free variables, ACA_0 proves that

$$\forall e' \exists e \forall m \forall X (\varphi(e', m, X) \leftrightarrow \tilde{\pi}(e, m, X)).$$

We can then formalise relative Turing computability in second-order arithmetic using $\tilde{\pi}$ by saying that a set $Y \subseteq \mathbb{N}$ is X -recursive and write $Y \leq_T X$ if there exists $e = \langle e_0, e_1 \rangle \in \mathbb{N}$ such that

$$Y = \{m : \tilde{\pi}(e_0, m, X)\} = \{m : \neg \tilde{\pi}(e_1, m, X)\}.$$

We say that e is the X -recursive index of Y , and we write

$$\text{Rec}^X = \{e \in \mathbb{N} : \forall m (\tilde{\pi}(e_0, m, X) \leftrightarrow \neg \tilde{\pi}(e_1, m, X))\}$$

to denote the set of all X -recursive indices. For any $e \in \text{Rec}^X$, let R_e^X denote the set defined by e . The superscript X will be omitted when $X = \emptyset$.

Definition 3.4.1 (Ordinal notations). Let $\mathcal{O}_+$ denote the set of indices $\alpha \in \mathbb{N}$ where $\alpha_0 \in \text{Rec}$, R_{α_0} is a weak linear ordering on the set

$$\text{field}(R_{\alpha_0}) = \{n \in \mathbb{N} : \exists m (\langle n, m \rangle \in R_{\alpha_0} \vee \langle m, n \rangle \in R_{\alpha_0})\}$$

and $\alpha_1 \in \text{field}(R_{\alpha_0})$. The canonical partial ordering on $\mathcal{O}_+$ is given by

$$\alpha \leq_{\mathcal{O}} \beta \quad :\Longleftrightarrow \quad \alpha_0 = \beta_0 \wedge \langle \alpha_1, \beta_1 \rangle \in R_{\alpha_0}.$$

We then let $\mathcal{O}$ denote the class of indices α such that $<_{\mathcal{O}}$ is a well-ordering on the initial segment $\{\beta \in \mathcal{O}_+ : \beta <_{\mathcal{O}} \alpha\}$, i.e.

$$\mathcal{O} = \{\alpha \in \mathcal{O}_+ : \neg \exists f : \mathbb{N} \rightarrow \mathcal{O}_+ (f(0) = \alpha \wedge \forall n f(n+1) <_{\mathcal{O}} f(n))\}$$

¹⁷We use $\tilde{\pi}$ since, strictly speaking, this formula has a different number of free variables than π considered in section 3.2.

Given any $\alpha \in \mathcal{O}$, we say its *order type* is precisely the order type¹⁸ of this initial segment

$$\text{otyp}(\alpha) = \text{otyp}(\{\beta \in \mathcal{O}_+ : \beta <_{\mathcal{O}} \alpha\}, \leq_{\mathcal{O}}).$$

Definition 3.4.2 (*H*-sets). The *Turing jump* of a set $X \subseteq \mathbb{N}$ is given as

$$\text{TJ}(X) = \{\langle e, m \rangle : \tilde{\pi}(e, m, X)\}.$$

For an ordinal notation α , the α -iterated *Turing jumps*, or as Simpson [43] calls it, an *H-set* for the ordinal α , is some set Y satisfying the following formula:

$$H(\alpha, Y) \quad :\Longleftrightarrow \quad \alpha \in \mathcal{O}_+ \wedge Y = (Y)_{<_{\mathcal{O}} \alpha} \wedge \forall \beta <_{\mathcal{O}} \alpha (Y)_{\beta} = \text{TJ}\left((Y)_{<_{\mathcal{O}} \beta}\right),$$

where the truncation of Y before α is

$$(Y)_{<_{\mathcal{O}} \alpha} = \left\{ \langle \beta, m \rangle : \beta <_{\mathcal{O}} \alpha \wedge m \in (Y)_{\beta} \right\}.$$

As in [43, Theorem VIII.3.15], ATR_0 suffices to prove $\forall \alpha \in \mathcal{O} \exists Y H(\alpha, Y)$. We additionally know that:

Proposition 3.4.3 (ACA_0 , [43, Lemma VIII.3.14]). *Given $\alpha, \beta \in \mathcal{O}$ and some Y such that $H(\alpha, Y)$, then the order types $\text{otyp}(\alpha)$ and $\text{otyp}(\beta)$ are comparable, and the comparison map is $\leq_T Y$.*

Thus, any $\alpha, \beta \in \mathcal{O}$ are comparable in ATR_0 , and we can verify that the comparison function is effectively computed by our weak *pca* Hyp:

Corollary 3.4.4 (ATR_0). *There exists $C \in \text{Hyp}$ such that for any $\alpha \in \mathcal{O}$, $\beta \in \mathbb{N}$, $C\alpha\beta \downarrow \in \{0, 1\}$ and*

$$C\alpha\beta = 1 \leftrightarrow \beta \in \mathcal{O} \wedge \text{otyp}(\beta) \leq \text{otyp}(\alpha).$$

Proof. By Lemma 3.2.1, there is $C \in \text{Hyp}$ such that for any $\alpha, \beta \in \mathbb{N}$, $C\alpha\beta \downarrow = d$ if and only if

- (a) either $d = 1$ and there exists Y such that $H(\alpha, Y)$, $\beta \in \mathcal{O}_+$ and there exists $e \in \text{Rec}^Y$ such that R_e^Y is an order-isomorphism between the set

¹⁸We will not fix a concrete object representing an ‘order type’ here in second-order arithmetic. Rather, given two well-orderings $\langle X, \leq_X \rangle$ and $\langle Y, \leq_Y \rangle$, we simply treat statements $\text{otyp}(X, \leq_X) = \text{otyp}(Y, \leq_Y)$ and $\text{otyp}(X, \leq_X) \leq \text{otyp}(Y, \leq_Y)$ as abbreviations for the respective assertions that corresponding comparison maps exist.

$\{\gamma \in \mathcal{O}_+ : \gamma <_{\mathcal{O}} \beta\}$ and an initial segment of $\{\gamma \in \mathcal{O}_+ : \gamma <_{\mathcal{O}} \alpha\}$;

- (b) or $d = 0$ and there exists Y such that $H(\alpha, Y)$ and the remainder of condition (a) does not hold.

It is easy to see that this definition is Σ_1^1 and precisely one of the two conditions must apply for any $\alpha \in \mathcal{O}$ since the corresponding H -set Y is necessarily unique by arithmetic transfinite induction, thus indeed $C\alpha\beta \downarrow \in \{0, 1\}$ for any $\alpha \in \mathcal{O}$ and $\beta \in \mathbb{N}$.

Now suppose that $C\alpha\beta = 1$, then $\beta \in \mathcal{O}$ because any strictly decreasing sequence below β will correspond to a strictly decreasing sequence below α through R_e^Y . Conversely, suppose that $\beta \in \mathcal{O}$ and $\text{otyp}(\beta) \leq \text{otyp}(\alpha)$, then Proposition 3.4.3 ensures that the comparison function is Y -recursive, so $C\alpha\beta$ must evaluate to 1 indeed. $\square$

Next, we consider a representation of all hyperarithmetic sets as the collection of indices

$$\mathcal{H} = \{a : a_0 \in \mathcal{O} \wedge \exists Y (H(a_0, Y) \wedge a_1 \in \text{Rec}^Y)\}.$$

We say that a set $X \subseteq \mathbb{N}$ is *hyperarithmetic* if there exists $a \in \mathcal{H}$ and $Y \subseteq \mathbb{N}$ such that $H(a_0, Y)$ and $X = R_{a_1}^Y$. We say that $a = \langle a_0, a_1 \rangle$ is a *hyperarithmetic index* of X .

We shall effectivise the Kleene–Souslin theorem [43, Theorem VIII.3.19], and state that the weak pca Hyp computes a function from S_2 to $\mathcal{H}$ that maps characteristic functions $\mathbb{N} \rightarrow \{0, 1\}$ to their corresponding hyperarithmetic sets, i.e.

Theorem 3.4.5 (ATR_0). *There exists $H \in \text{Hyp}$ such that*

- (i) *for each $e \in S_2$, $He \downarrow \in \mathcal{H}$ is a hyperarithmetic index of $\{n : en = 1\}$;*
- (ii) *for $d, e \in S_2$, $Hd = He$ if and only if $d \equiv e$.*

Proof. We will replicate some key constructions in Simpson’s proof, and then show that they can be effectively specified by a Σ_1^1 -formula. For any $e \in S_2$, it is easy to see that $en = 1$ if and only if $\exists X \pi(e_0, e_1, n, 1, X)$, which is Σ_1^1 . Observe that $en = 1$ also if and only if $en \neq 0$, if and only if

$$\forall X \neg \pi(e_0, e_1, n, 0, X),$$

which is Π_1^1 ; so $en = 1$ is a Δ_1^1 -assertion.

By Kleene normal form theorem for Π_1^1 -formulae ([43, Theorem V.1.4]), there exists a Δ_0^0 -formula $\theta(e, n, \tau)$ such that ACA_0 proves

$$\forall e \in S_2 \forall n (en = 1 \leftrightarrow \forall f \exists k \neg \theta(e, n, f \upharpoonright k))$$

where f ranges over second-order objects representing functions $\mathbb{N} \rightarrow \mathbb{N}$, and $f \upharpoonright k = \langle f(0), \dots, f(k-1) \rangle$ is a finite tuple encoded as a first-order object. We then consider recursive trees

$$T_n^{(e)} = \{ \tau \in \mathbb{N}^{<\mathbb{N}} : \forall k \leq |\tau| \theta(e, n, \tau \upharpoonright k) \}$$

and let $\text{KB}_n^{(e)} = \text{KB}(T_n^{(e)})$ be the Kleene–Brouwer ordering of $T_n^{(e)}$, i.e. $\sigma \leq_{\text{KB}} \tau$ if either $\sigma \sqsupseteq \tau$ or that σ has a smaller component at the first index where the two differ (cf. [43, Section V.1]). It follows that

$$\begin{aligned} en = 1 & \quad \text{iff} \quad \forall f \exists k \neg \theta(e, n, f \upharpoonright k), \\ & \quad \text{iff} \quad T_n^{(e)} \text{ has no path,} \\ & \quad \text{iff} \quad \text{KB}_n^{(e)} \text{ is a recursive well-ordering.} \end{aligned}$$

Now, we work in ATR_0 where $\forall \alpha \in \mathcal{O} \exists Y H(\alpha, Y)$. Then Simpson’s argument applies to justify that the collection of well-orderings $\text{KB}_n^{(e)}$ (for all $n \in \mathbb{N}$, given a fixed e) has bounded order type in $\mathcal{O}$ — otherwise, $\mathcal{O}$ can be defined precisely as the Σ_1^1 -class of $\beta \in \mathcal{O}_+$ such that $C\alpha\beta = 1$ for some $\alpha = \text{KB}_n^{(e)}$ where $n \in \mathbb{N}$ and $en = 1$, contradicting [43, Lemma VIII.3.4].

Let $\alpha \in \mathcal{O}$ be such a bound and $\Omega \subseteq \mathbb{N}$ be such that $H(\alpha, \Omega)$. By Proposition 3.4.3, the order types of any initial segment of $\text{KB}_n^{(e)}$ and any $\beta \leq_{\mathcal{O}} \alpha$ will all be comparable via comparison maps $\leq_T \Omega$, which means for any such well-orderings A, B , a comparison statement $\text{otyp}(A) \leq \text{otyp}(B)$ will be an arithmetic formula ‘there exists $a \in \mathbb{N}$ such that R_a^Ω is an order isomorphism between A and an initial segment of B ’.

Therefore, by Δ_1^1 -comprehension (cf. [43, Theorem V.5.1]),

$$\Gamma = \left\{ \beta \leq_{\mathcal{O}} \alpha : \forall n \left(en = 1 \rightarrow \text{otyp}(\text{KB}_n^{(e)}) \leq \text{otyp}(\beta) \right) \right\}$$

is a set and we can find $\lambda = \min \Gamma$. For any $\gamma <_{\mathcal{O}} \lambda$, there must exist $n \in \mathbb{N}$ such that $en = 1$ and $\text{otyp}(\gamma) < \text{otyp}(\text{KB}_n^{(e)})$. In fact, these two characteristics determine λ uniquely up to order isomorphism, i.e. we can prove the following lemma that:

Lemma. Suppose that $\rho \in \mathcal{O}_+$ satisfies both

- (i) $\text{otyp}\left(\text{KB}_n^{(e)}\right) \leq \text{otyp}(\rho)$ for any n such that $en = 1$,
- (ii) and for any $\gamma <_{\mathcal{O}} \rho$, there exists $n \in \mathbb{N}$ such that $en = 1$ and $\text{otyp}(\gamma) < \text{otyp}\left(\text{KB}_n^{(e)}\right)$,

then $\rho \in \mathcal{O}$ and $\text{otyp}(\rho) = \text{otyp}(\lambda)$.

Proof. Firstly, assume that $\rho \notin \mathcal{O}$, then there exists a strictly decreasing sequence $f : \mathbb{N} \rightarrow \mathcal{O}_+$ such that $f(0) = \rho$. We have $f(1) <_{\mathcal{O}} \rho$, hence we can consider some order-preserving map $g : \{\gamma : \gamma <_{\mathcal{O}} f(1)\} \rightarrow \text{KB}_n^{(e)}$. Now $m \mapsto g(f(m+2))$ is a descending sequence in $\text{KB}_n^{(e)}$, a contradiction.

Therefore, we must have $\rho \in \mathcal{O}$. We then know in ATR_0 that the order types of ρ and λ are comparable, but it is easy to verify that we can have neither $\text{otyp}(\rho) > \text{otyp}(\lambda)$ nor $\text{otyp}(\rho) < \text{otyp}(\lambda)$. $\square$

Finally, given any $\lambda \in \mathcal{O}$, we can remove the ambiguity around the order-isomorphic ordinal notations by requiring that λ is minimal among all order-isomorphic elements in $\mathcal{O}_+$, with respect to the natural ordering on $\mathbb{N}$. Such a minimum exists by Δ_0^0 -induction because, using some $Y \subseteq \mathbb{N}$ satisfying $H(\lambda, Y)$ as a parameter, the set of elements in $\mathcal{O}_+$ order-isomorphic to λ is an arithmetic set. The rest of Simpson's argument in [43, Theorem VIII.3.19] applies to show that if $H(\lambda + \omega, Y)$, then $\{n : en = 1\} \leq_T Y$.

Therefore, by Lemma 3.2.1, we can stipulate $He \downarrow = a$ when there exists $\alpha \in \mathcal{O}_+$ and $Y \subseteq \mathbb{N}$ satisfying all of the following:

- (i) $H(a_0, Y)$ and¹⁹ $\text{otyp}(a_0) = \text{otyp}(\alpha + \omega)$;
- (ii) for any $n \in \mathbb{N}$, $en = 1 \rightarrow \text{otyp}\left(\text{KB}_n^{(e)}\right) \leq \text{otyp}(\alpha)$;
- (iii) for any $\beta <_{\mathcal{O}} \alpha$, there exists $n \in \mathbb{N}$ such that $en = 1$ and $\text{otyp}(\beta) < \text{otyp}\left(\text{KB}_n^{(e)}\right)$;
- (iv) for any $\gamma \in \mathcal{O}_+$, if there is a Y -recursive order-isomorphism between γ and a_0 , then $\gamma \geq a_0$;
- (v) $a_1 \in \text{Rec}^Y$ and for any $n \in \mathbb{N}$, $en = 1 \leftrightarrow n \in R_{a_1}^Y$;

¹⁹Here we fix $\alpha + \omega$ to mean the following specific recursive well-ordering in $\mathcal{O}$ which can be computed uniformly from α : order the set $\{2\beta + 2 : \beta <_{\mathcal{O}} \alpha\}$ by

$$2\beta_1 + 2 < 2\beta_2 + 2 \quad :\Longleftrightarrow \quad \beta_1 <_{\mathcal{O}} \beta_2,$$

and append the sequence $1, 3, 5, 7, \dots, 0$ in this order. Then the initial segment before 0 has order type $\alpha + \omega$.

(vi) for any $b \in \text{Rec}^Y$, if $n \in R_{a_1}^Y \leftrightarrow n \in R_b^Y$ holds for all $n \in \mathbb{N}$, then $b \geq a_1$.

Using the fact that $en = 1$ is Δ_1^1 and the comparison maps for the concerned order types are all Y -recursive, this definition can be written as a Σ_1^1 -formula, and conditions (iii), (vi) ensure that $a = \langle a_0, a_1 \rangle$ satisfying them is necessarily unique. Thus for any $e \in S_2$, we see that $He \downarrow$.

To verify that $Hd = He$ if and only if $d \equiv e$, simply observe that the value of He is uniquely determined by the set $\{n : en = 1\}$. $\square$

The (rough) inverse is much easier and can be constructed in ACA_0 :

Proposition 3.4.6 (ACA_0). *There exists²⁰ $H^{-1} \in \text{Hyp}$ such that, for any $a \in \mathcal{H}$ and any $Y \subseteq \mathbb{N}$ satisfying $H(a_0, Y)$, we have $H^{-1}a \downarrow \in S_2$ and*

$$\forall n (H^{-1}an = 1 \leftrightarrow n \in R_{a_1}^Y).$$

Proof. By Lemma 3.2.1, pick $H^{-1} \in \text{Hyp}$ such that $H^{-1}an \downarrow = i$ if and only if

$$\exists Y (H(a_0, Y) \wedge ((\varphi(a, n, Y) \wedge i = 1) \vee (\neg \varphi(a, n, Y) \wedge i = 0)))$$

where $\varphi(a, n, Y) = a_1 \in \text{Rec}^Y \wedge n \in R_{a_1}^Y$ is arithmetic. $\square$

Do note that H and H^{-1} are inverses of each other only up to extensional equivalences, i.e. for any $e \in S_2$, we have $H^{-1}(He) \equiv e$, and for any $a \in \mathcal{H}$, a and $H(H^{-1}a)$ are hyperarithmetic indices for the same set.

Realising the axioms of GWO

By Lemma 3.2.1, let $R \in \text{Hyp}$ be such that $Ra \downarrow = d$ if and only if there exists $Y \subseteq \mathbb{N}$ such that $H(a_0, Y)$ and either $a_1 \in \text{Rec}^Y$ and $d = 1$ or $a_1 \notin \text{Rec}^Y$ and $d = 0$, so that for any $a \in \mathbb{N}$, if $a_0 \in \mathcal{O}$ then $Ra \downarrow \in \{0, 1\}$ and

$$Ra = 1 \leftrightarrow a \in \mathcal{H}.$$

Let $C^* \in \text{Hyp}$ be the function

$$\lambda ab. \begin{cases} 1 & \text{if } C(\mathbf{p}_0 a)(\mathbf{p}_0 b) = 1 \wedge (C(\mathbf{p}_0 b)(\mathbf{p}_0 a) = 0 \vee b < a) \wedge Rb = 1, \\ 0 & \text{otherwise,} \end{cases}$$

²⁰Formally speaking, $H \in \text{Hyp}$ is only a natural number. But I feel it is understandable here to abuse notation and let H^{-1} denote another natural number, which computes roughly the inverse of what H computes.

defined using C from Corollary 3.4.4. To be pedantic, observe that when $\mathbf{p}_0 a = a_0 \in \mathcal{O}$, $C(\mathbf{p}_0 a)(\mathbf{p}_0 b) \downarrow$ for any $b \in \mathbb{N}$; and $C(\mathbf{p}_0 a)(\mathbf{p}_0 b) = 1$ implies that $\mathbf{p}_0 b = b_0 \in \mathcal{O}$, in which case $C(\mathbf{p}_0 b)(\mathbf{p}_0 a)$ and Rb will also converge, hence they will be safe to evaluate when guarded behind the lazy $\mathbf{d}_{\mathbb{N}}^*(C(\mathbf{p}_0 a)(\mathbf{p}_0 b))1$. Thus, as long as $a \in \mathcal{H}$, we have $C^*ab \downarrow$ for any $b \in \mathbb{N}$, and it is easy to observe that $C^*ab = 1$ implies $b \in \mathcal{H}$.

Proposition 3.4.7 (ATR₀). *The relation $\prec$ on $\mathcal{H}$ given by*

$$b \prec a \quad :\Longleftrightarrow \quad C^*ab = 1$$

is a well-ordering.

Proof. $\prec$ is a linear ordering simply because any two elements in $\mathcal{O}$ are comparable, and $b \in \mathcal{H}$ ensures that $Rb = 1$.

Suppose $\prec$ is not a well-ordering, i.e. we have a strictly decreasing function $f : \mathbb{N} \rightarrow \mathcal{H}$. Then $f_0 : \mathbb{N} \rightarrow \mathcal{O}$ given by $f_0(n) = f(n)_0$ satisfies

$$\text{otyp}(f_0(0)) \geq \text{otyp}(f_0(1)) \geq \cdots$$

We have $\text{otyp}(f_0(0)) \geq \text{otyp}(f_0(n))$ for any $n \in \mathbb{N}$, so there must be a unique $\gamma_n \leq_{\mathcal{O}} f_0(0)$ such that $\text{otyp}(f_0(n)) = \text{otyp}(\gamma_n)$. Now, fix $\Omega \subseteq \mathbb{N}$ such that $H(f_0(0), \Omega)$, then all order type statements concerning $f_0(n)$ will be arithmetic relative to Ω . Thus, by arithmetic comprehension, there exists a function $f' : n \mapsto \gamma_n$ and we immediately have $f_0(0) = f'(0) \geq_{\mathcal{O}} f'(1) \geq_{\mathcal{O}} \cdots$. Since $f_0(0) \in \mathcal{O}$, f' must stabilise after some $N \in \mathbb{N}$, implying that $\text{otyp}(f_0(m)) = \text{otyp}(f_0(n))$ for any $m, n > N$.

Then, for any $m > N$, we will have $f(m) \succ f(m+1)$ yet $\text{otyp}(f(m)_0) = \text{otyp}(f(m+1)_0)$, thus we must have $f(m) > f(m+1)$ by the definition of C^* . Thus, f restricted to $\{m : m > N\}$ is a strictly decreasing sequence of natural numbers. This contradicts Δ_0^0 -induction. Therefore, such f cannot exist and $\prec$ is indeed a well-ordering. $\square$

Now, move to the theory ATR (with unrestricted induction on the natural numbers, which is only needed for the induction and recursion schemes of CM) and note that this is an extension of Σ_1^1 -AC (cf. [43, Theorem V.8.3]). We reuse the realisability conditions in section 3.2, but require that the interpretation q for $\prec$ is chosen as

$$q = \lambda e f. C^*(Hf)(He).$$

Theorem 3.4.5 guarantees that $Hd = He \leftrightarrow d \equiv e$, and ensures that q satisfies

$$\forall e, e', f, f' \in S_2 (e \equiv f \wedge e' \equiv f' \rightarrow qee' = qff').$$

The axioms (W1) stating that $\prec$ is a linear ordering are simply propositional compositions of atomic formulae of the form $X \prec Y$ and $X = Y$. Since their truths are computable²¹ in Hyp for any assignments from S_2 and q indeed computes a well-ordering on S_2 (up to extensional equivalence) by Proposition 3.4.7, these axioms are realised by any realiser that is total on the appropriate inputs. (W3) is another easy case:

Lemma 3.4.8. *The axiom of countable initial segments (W3) is realised:*

$$\forall X \exists Z \forall Y (Y \prec X \rightarrow \exists n Y = (Z)_n).$$

Proof. Consider any $x \in S_2$, then for any $y \in \mathbb{N}$, $C^*(Hx)y \downarrow$ and it evaluates to 1 if and only if $y \in \mathcal{H}$ and $y \prec Hx$ as defined in Proposition 3.4.7. Thus, we simply need some Z containing the second-order objects $H^{-1}y \in S_2$ for all y such that $C^*(Hx)y = 1$ (up to extensional equivalence).

Formally, we shall define

$$z = \lambda n. \mathbf{d}_{\mathbb{N}}^*(C^*(Hx)(\mathbf{p}_0 n))1(H^{-1}(\mathbf{p}_0 n)(\mathbf{p}_1 n))0.$$

For any $y \in S_2$ such that $qyx = 1$, we see that

$$z(\mathbf{p}(Hy)n) = H^{-1}(Hy)n = yn$$

by extensional equivalence between y and $H^{-1}(Hy)$, thus it is easy to check that $\mathbf{p}(Hy)(\lambda m. (\mathbf{p}(\mathbf{p}(Hy)m)d))$ realises $\exists n y = (z)_n$ interpreted as

$$\exists n \forall m \exists k (k = \langle n, m \rangle \wedge (m \in_1 y \leftrightarrow k \in_1 z)),$$

where d is the uninteresting realiser for the entire quantifier-free subformula inside. □

Now, ATR does not suffice to realise the transfinite induction scheme (W2)

²¹To decide whether $e = f$ should be realised, i.e. whether $e \equiv f$ for $e, f \in S_2$, we can either observe that

$$\forall X \forall Y (X = Y \vee \neg X = Y)$$

follows from numerical omniscience and use the mechanisms in Lemma 3.2.6, or just conveniently check whether $He = Hf$.

for formula φ of arbitrary complexity, since the base theory cannot support transfinite induction on well-orderings for any second-order formulae (cf. footnote 14 on page 28 for the quantifier complexity of the realisability conditions). For that purpose, we require the much stronger system BI — also known as $\Pi_\infty^1\text{-TI}_0$ in Simpson [43] — which adds to ACA_0 the axiom scheme of *bar induction*,

$$\forall X \forall <_X (\text{WO}(X, <_X) \rightarrow \forall x \in X (\forall y <_X x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \in X \varphi(x))$$

for any formula $\varphi(x)$, where $\text{WO}(X, <_X)$ denotes the Π_1^1 -formula asserting that $<_X$ is a linear ordering on X and there exists no strictly decreasing function $f : \mathbb{N} \rightarrow X$.

It does not matter how much induction on the natural numbers we have inside BI, since Δ_0^0 -induction suffices to verify $\text{WO}(\mathbb{N}, <)$, and thus bar induction already implies unrestricted induction on $\mathbb{N}$. Additionally, by [43, Corollary VII.2.19], BI implies ATR, so the existing axioms of CM, (W1) and (W3) are already realised in BI.

In order to realise (W2), we need the following lemma:

Lemma 3.4.9 ($\Sigma_1^1\text{-AC}$). *For any $d \in \text{Hyp}$ and $e, f \in S_2$ such that $e \equiv f$, if $d \Vdash \varphi[e/X]$ for some formula $\varphi(X)$ (possibly with other parameters), then also $d \Vdash \varphi[f/X]$.*

Specifically, assuming ATR, $d \Vdash \varphi[e/X]$ if and only if $d \Vdash \varphi[H^{-1}(He)/X]$.

Proof. This simply amounts to the fact that when φ is atomic, the realiser d is irrelevant by our stipulations and the statement is equivalent to saying that $\varphi[e/X]$ is realised if and only if $\varphi[f/X]$ is. Thus, this holds for any formula $\varphi(X)$ by induction on the complexity of φ . $\square$

Proposition 3.4.10 (BI). *The axiom of unrestricted transfinite induction (W2) is realised for any formula $\varphi(X)$:*

$$\forall X (\forall Y (Y \prec X \rightarrow \varphi(Y)) \rightarrow \varphi(X)) \rightarrow \forall X \varphi(X).$$

Proof. We shall define

$$h = \lambda f. \mathbf{r}(\lambda a g. f(H^{-1}a)(\lambda z d. g(Hz))),$$

and we claim that $\lambda \vec{x} f y. h f(Hy)$ realises the axiom, where $\vec{x}$ enumerate the

free variables in it.

Fix some $f \Vdash \forall X (\forall Y (Y \prec X \rightarrow \varphi(Y)) \rightarrow \varphi(X)) [\vec{u}/\vec{x}]$ and any $y \in S_2$. We consider the condition

$$\psi(a) \quad :\Longleftrightarrow \quad a \in \mathcal{H} \wedge a \preceq Hy \wedge a = H(H^{-1}a).$$

Note that $a \prec Hy$ if and only if $C^*(Hy)a = 1$, which already implies $a \in \mathcal{H}$ and that $H(H^{-1}a) \downarrow$, thus $\psi(a)$ is equivalent to a Σ_1^1 -formula. On the other hand, given $H(H^{-1}a) \downarrow$, we have $a \neq H(H^{-1}a)$ if and only if $\exists b \neq a H(H^{-1}a) = b$. So $\neg\psi(a)$ is equivalent to

$$(a \neq Hy \wedge C^*(Hy)a = 0) \vee (C^*(Hy)a = 1 \wedge \exists b \neq a H(H^{-1}a) = b),$$

which is also Σ_1^1 . Therefore, $\psi(a)$ is Δ_1^1 and by Δ_1^1 -comprehension (which is a consequence of ATR_0 , cf. [43, Theorem V.5.1], and thus holds in BI), we can define the set

$$W = \{a : \psi(a)\} = \{a \in \mathcal{H} : a \preceq Hy, a = H(H^{-1}a)\}$$

Observe additionally that for any $z \in S_2$, $H(H^{-1}(Hz)) = Hz$ by Theorem 3.4.5 since $H^{-1}(Hz) \equiv z$, thus $Hz \in W$ as long as $Hz \preceq Hy$.

By Proposition 3.4.7, the set W is well-ordered by $\prec$ and we shall prove by bar induction that for any $a \in W$,

$$hfa \downarrow \Vdash \varphi[H^{-1}a/X, \vec{u}/\vec{x}].$$

Consider any $z \in S_2$ such that $qz(H^{-1}a) = 1$, i.e. $Hz \prec H(H^{-1}a) = a$. By the inductive hypothesis, we have

$$hf(Hz) \downarrow \Vdash \varphi[H^{-1}(Hz)/X, \vec{u}/\vec{x}],$$

which also realises $\varphi[z/X, \vec{u}/\vec{x}]$ by Lemma 3.4.9. This means that

$$\lambda zd. hf(Hz) \Vdash \forall Y (Y \prec X \rightarrow \varphi(Y)) [H^{-1}a/X, \vec{u}/\vec{x}].$$

It follows that

$$f(H^{-1}a)(\lambda zd. hf(Hz)) \downarrow \Vdash \varphi[H^{-1}a/X, \vec{u}/\vec{x}]$$

and thus so does $hfa \downarrow$ by the recursion theorem. Therefore, by bar induction,

it follows that

$$hf(Hy)\downarrow \Vdash \varphi[H^{-1}(Hy)/X, \vec{u}/\vec{x}],$$

thus the same realiser also realises $\varphi[y/X, \vec{u}/\vec{x}]$ by Lemma 3.4.9. $\square$

It follows that all the axioms of CM + GWO are interpreted in the given realisability model in BI, hence immediately:

Theorem 3.4.11. *Let φ be any theorem of CM + GWO, then BI proves there exists $d \in \text{Hyp}$ such that $d \Vdash \varphi$.*

An ordinal analysis of CM + GWO

To give a lower bound for the proof-theoretic strength of CM + GWO, we say that a formula $\varphi(X, x)$ in the language of CM is a *strictly positive first-order formula*, possibly with other free (first-order) variables, if it is built up according to the following rules:

- (1) φ can contain any atomic formulae in the language of PA, as well as the atomic formula $t \in_1 X$ where t is an arbitrary first-order term;
- (2) φ can contain the propositional constant ‘bottom’, $\perp$, which may be interpreted as $\neg 0 = 1$;
- (3) φ can contain connectives $\wedge, \vee, \rightarrow$, with the restriction that X cannot occur on the left of the connective $\rightarrow$;
- (4) φ can contain quantifiers $\forall y$ and $\exists y$ where y is any first-order variable.

Let $\theta(x)$ be any formula, and we let $\varphi(\theta, x)$ denote the result of substituting every occurrence of $t \in_1 X$ inside $\varphi(X, x)$ by $\theta(t)$, with appropriate renaming so that any other free variable in $\theta(x)$ is not bound by any quantifiers in φ .

We shall prove the following:

Theorem 3.4.12 (Inductive definitions). *For any strictly positive first-order formula $\varphi(X, x)$, there exists $\theta_\varphi(x)$ such that*

$$\text{CM} + \text{GWO} \vdash \forall x (\varphi(\theta_\varphi, x) \rightarrow \theta_\varphi(x)),$$

and also for any formula $\eta(x)$

$$\text{CM} + \text{GWO} \vdash \forall x (\varphi(\eta, x) \rightarrow \eta(x)) \rightarrow \forall x (\theta_\varphi(x) \rightarrow \eta(x)).$$

This will involve a few lemmata:

Lemma 3.4.13 (Positivity). *For any formulae $\theta(x), \eta(x)$,*

$$\text{CM} \vdash \forall x (\theta(x) \rightarrow \eta(x)) \rightarrow \forall x (\varphi(\theta, x) \rightarrow \varphi(\eta, x)).$$

This trivially implies $X \subseteq Y \rightarrow \forall x (\varphi(X, x) \rightarrow \varphi(Y, x))$ as well.

Proof. This follows from an easy induction on the complexity of $\varphi(X, x)$. $\square$

Lemma 3.4.14. *For any formula $\theta(x)$,*

$$\text{CM} \vdash \forall x (\varphi(\theta, x) \rightarrow \exists X (\forall z \in_1 X \theta(z) \wedge \varphi(X, x)))$$

Proof. We show this by induction on the complexity of $\varphi(X, x)$:

Firstly, if $\varphi(X, x)$ is an atomic formula not of the form $t \in_1 X$, then $\varphi(\theta, x) \rightarrow \varphi(\emptyset, x)$ trivially; if $\varphi(X, x)$ is $t \in_1 X$, then $\varphi(\theta, x) \rightarrow \forall z \in_1 \{t\} \theta(z) \wedge \varphi(\{t\}, x)$, where t is any first-order term containing the variable x .

The case where φ is $\perp$ is vacuously true.

Now, suppose that φ is of the form $\psi \wedge \chi$ and $\varphi(\theta, x)$ holds, then by the inductive hypothesis we have X and Y such that

$$\forall z \in_1 X \cup Y \theta(z) \wedge \psi(X, x) \wedge \chi(Y, x).$$

It follows from Lemma 3.4.13 that we also have $\psi(X \cup Y, x) \wedge \chi(X \cup Y, x)$.

Suppose φ is of the form $\psi \rightarrow \chi$ and $\varphi(\theta, x)$ holds, we know from stipulation that $\psi(x)$ must not contain X . By the inductive hypothesis, we have

$$\psi(x) \rightarrow \exists X (\forall z \in_1 X \theta(z) \wedge \chi(X, x)).$$

Since $\psi(x)$ is arithmetic, hence decidable, we can take Y to be X above if $\psi(x)$ holds and $\emptyset$ otherwise. Then we always have $\forall z \in_1 Y \theta(z)$, and we also have $\psi(x) \rightarrow \chi(Y, x)$.

Suppose φ is of the form $\forall y \psi$ and $\varphi(\theta, x)$ holds, then by the inductive hypothesis

$$\forall y \exists X (\forall z \in_1 X \theta(z) \wedge \psi(X, x, y)).$$

By the recursion axiom of CM, we can find Z such that

$$\forall y \left(\forall z \in_1 (Z)_y \theta(z) \wedge \psi((Z)_y, x, y) \right).$$

Take $Y = \bigcup_y (Z)_y$, then $\forall z \in_1 Y \theta(z)$ and also $\forall y \psi(Y, x, y)$ by Lemma 3.4.13.

Finally, the cases where φ is of the form $\psi \vee \chi$ or $\exists y \psi$ are trivial. $\square$

Proof of Theorem 3.4.12. Let a formula $\psi(X, Z, S)$ assert that both of the following hold:

$$(i) \quad \forall Y \preceq X \exists n Y = (Z)_n;$$

$$(ii) \quad \forall n ((Z)_n \preceq X \rightarrow \forall x (x \in_1 (S)_n \leftrightarrow \varphi(\bigcup \{(S)_m : (Z)_m \prec (Z)_n\}, x))).$$

Observe that if $(S)_m$ is given for all m such that $(Z)_m \prec (Z)_n$, then the unique $(S)_n$ can be constructed by decidable comprehension as φ is arithmetic. Thus, by transfinite induction on all second-order objects, it is easy to show that for each X , a pair Z, S satisfying $\psi(X, Z, S)$ exists and is unique up to matched permutations, i.e. for any n, m ,

$$\psi(X, Z, S) \wedge \psi(X, Z', S') \wedge (Z)_n \preceq X \wedge (Z)_n = (Z')_m \rightarrow (S)_n = (S')_m.$$

Now, we denote

$$\theta_\varphi(x) \quad :\Longleftrightarrow \quad \exists X, Z, S (\psi(X, Z, S) \wedge \exists n (X = (Z)_n \wedge x \in_1 (S)_n)).$$

We will verify that CM + GWO proves the desired properties of $\theta_\varphi(x)$: firstly, suppose that $\varphi(\theta_\varphi, x)$ holds for some x . By Lemma 3.4.14, we can find A satisfying

$$\forall x \in_1 A \theta_\varphi(x) \wedge \varphi(A, x).$$

Using the recursion axiom, the former conjunct will imply the existence of families X, Z, S such that

$$\forall x \in_1 A (\psi((X)_x, (Z)_x, (S)_x) \wedge \exists n ((X)_x = ((Z)_x)_n \wedge x \in_1 ((S)_x)_n)).$$

By Lemma 3.3.3, we can find some Y such that $\forall x \in_1 A (X)_x \prec Y$. Let Z^*, S^* be such that $\psi(Y, Z^*, S^*)$ holds, then it follows that

$$A \subseteq \bigcup \{(S^*)_m : (Z^*)_m \prec Y\}.$$

Now $\varphi(A, x)$ and Lemma 3.4.13 will imply that $x \in_1 (S^*)_n$ where $(Z^*)_n = Y$,

and thus $\theta_\varphi(x)$.

Secondly, suppose that for some formula $\eta(x)$, $\forall x (\varphi(\eta, x) \rightarrow \eta(x))$ holds. In order to have $\forall x (\theta_\varphi(x) \rightarrow \eta(x))$, it suffices to fix X, Z, S satisfying $\psi(X, Z, S)$ and show by transfinite induction on Y that

$$\forall Y (Y \preceq X \rightarrow \forall n (Y = (Z)_n \rightarrow \forall x \in_1 (S)_n \eta(x))).$$

Here the inductive hypothesis implies

$$\forall x \in_1 \bigcup \{(S)_m : (Z)_m \prec Y\} \eta(x),$$

thus for any $x \in_1 (S)_n$, i.e. any x satisfying $\varphi(\bigcup \{(S)_m : (Z)_m \prec Y\}, x)$, Lemma 3.4.13 would imply $\varphi(\eta, x)$ and thus $\eta(x)$ as required. It follows by transfinite induction that $X = (Z)_n$ implies $\forall x \in_1 (S)_n \eta(x)$, and thus $\theta_\varphi(x)$ implies $\eta(x)$. $\square$

What we have shown is the existence of first-order inductive definitions in $\text{CM} + \text{GWO}$. Since Heyting arithmetic HA is trivially a subtheory of CM , Theorem 3.4.12 implies that $\text{CM} + \text{GWO}$ interprets the intuitionistic²² first-order theory of inductive definitions ID_1^i .

The proof-theoretic ordinal of ID_1^i is computed in [7] as $|\text{ID}_1^i| = \vartheta_{\varepsilon_{\Omega+1}}(0) = |\text{BI}|$, the Bachmann–Howard ordinal. Here Ω is treated as a formal name for the first uncountable ordinal ω_1 , and ϑ_α refers to any of the variants of ordinal collapse functions²³ in Buchholz [6], Schütte [42] or Pohlers [28]. Since $\text{CM} + \text{GWO}$ can be realised in BI , it is an immediate corollary that:

Corollary 3.4.15. $\text{CM} + \text{GWO} \equiv_{\text{Con}} \text{BI}$. *In particular, $|\text{CM} + \text{GWO}| = |\text{BI}| = \vartheta_{\varepsilon_{\Omega+1}}(0)$.*

3.5 Indispensability of countable initial segments

In this section, we use the realisability model above to showcase one key distinction between the global well-ordering axioms in the semi-intuitionistic system

²²Even though $\text{CM} \vdash \text{PA}$, we can only interpret the intuitionistic part here because we are translating the first-order predicate symbol I_φ for the least fixed point of φ into a non-decidable formula $\theta_\varphi(x)$.

²³These notations have minor differences, but they catch up at key positions like the Bachmann–Howard ordinal. Their differences are too intricate to be relevant throughout this thesis. For more detail, see discussions in [28, section 7] or the closing remarks in [29].

CM and the more traditional cases of choice principles in classical systems: that the axiom of countable initial segments (W3) is indispensable for the proof-theoretic strength of the system.

The readers may contrast this with, for example, classical Kripke–Platek set theory²⁴ KP, where Gödel’s constructible universe L is also a model of KP satisfying ‘there exists a global well-ordering’. Let ω_1^L denote what L believes to be the first uncountable ordinal — if it exists, or the class of all (necessarily countable) ordinals otherwise — then it is easy to verify

$$\text{KP} \vdash (\Delta_0\text{-collection})^{\omega_1^L}.$$

Additionally, by Gödel’s condensation lemma, for any $\alpha \in \omega_1^L$, any bijection $f : \alpha \rightarrow \omega$ in L must also lie in $L_{\omega_1^L}$, as f (as a set) is hereditarily countable. Thus $L_{\omega_1^L}$ is a model of KP satisfying the refined claim ‘there exists a global well-ordering whose every initial segment is countable’. This shows that adding the requirement of countable initial segments does not increase the proof-theoretic strength of the well-ordering assertions over KP.

This truncation cannot be done in CM however, since with a lack of the law of excluded middle, we can no longer classify the initial segments of the global well-ordering as either a countable one or an extension of all countable initial segments. Moreover, the fact that bounded second-order quantifiers, i.e. $\forall Y \prec X$ and $\exists Y \prec X$, are decidable, which is crucial for the applications of the well-ordering axioms we considered in section 3.3, relies critically on the countable initial segment assumption.

Here, we shall support these informal observations by demonstrating that the full system CM + GWO is indeed strictly stronger than the fragment CM + (W1) + (W2) without the axiom of countable initial segments. Let us work in ATR and again consider the realisability conditions given in section 3.2, where the interpretation of the well-ordering $\prec$ is now chosen as

$$q' = \lambda e f. \begin{cases} 1 & \text{if } He < Hf, \\ 0 & \text{otherwise.} \end{cases}$$

Recall from Theorem 3.4.5 that for $e, f \in S_2$, $He = Hf$ if and only if $e \equiv f$,

²⁴In this thesis, we shall use KP to denote the version of Kripke–Platek set theory that contains the axiom of infinity, which is sometimes referred to as $\text{KP}\omega$ in literature. The hereditarily finite fragment satisfying KP without infinity is too weak, in comparison to other theories assessed in this thesis, to be relevant here.

thus q' satisfies the extensionality requirement that

$$\forall e, e', f, f' \in S_2 (e \equiv f \wedge e' \equiv f' \rightarrow q'ee' = q'ff').$$

Once again, we see that the axioms (W1) stating that $\prec$ is a linear ordering are interpreted as propositional compositions of numerical inequalities in the base theory, thus realised by any realiser that is total on the appropriate inputs. We also have:

Proposition 3.5.1 (ATR). *Using the interpretation q' , the axiom of unrestricted transfinite induction (W2) is realised:*

$$\forall X (\forall Y (Y \prec X \rightarrow \varphi(Y)) \rightarrow \varphi(X)) \rightarrow \forall X \varphi(X).$$

Proof. Let $\vec{x}$ enumerate the free variables in the axiom and consider some $f \Vdash \forall X (\forall Y (Y \prec X \rightarrow \varphi(Y)) \rightarrow \varphi(X)) [\vec{u}/\vec{x}]$. By recursion, let

$$h = \lambda f. \mathbf{r}(\lambda a g. f(H^{-1}a)(\lambda y e. g(Hy))).$$

We show by induction on $\mathbb{N}$ that for any $a \in \mathcal{H}$ satisfying $a = H(H^{-1}a)$,

$$hfa \downarrow \Vdash \varphi(X) [H^{-1}a/X, \vec{u}/\vec{x}].$$

This is because the inductive hypothesis and Lemma 3.4.9 together imply that for any $y \in S_2$ such that $Hy < H(H^{-1}a) = a$ (where we always have $H(H^{-1}(Hy)) = Hy$), $hf(Hy) \downarrow \Vdash \varphi(X) [y/X, \vec{u}/\vec{x}]$, thus the (recursive) invocation of the realiser f would yield the desired result.

Therefore, $\lambda \vec{x} f y. hf(Hy)$ realises the axiom, by Lemma 3.4.9 again. $\square$

It follows that every theorem of $\text{CM} + (\text{W1}) + (\text{W2})$ is realised in ATR in this set-up. As argued in [21], the proof-theoretic strength of ATR is Γ_{ε_0} , much weaker than the full theory $\text{CM} + \text{GWO}$ at the Bachmann–Howard ordinal.

As a closing remark, we shall comment that, in fact, the only condition necessitating the use of hyperarithmetic indices and thus the developments in ATR_0 here is the fact that second-order equality in the language of CM is extensional, thus the ordering $\prec$ must respect extensional equivalences, i.e. satisfy axiom (8) from our definition of CM in section 3.1, in order to be a linear ordering.

Observe that, without extensionality requirements, the following much simpler

ordering would suffice:

$$q^\dagger = \lambda ef. \begin{cases} 1 & \text{if } e < f, \\ 0 & \text{otherwise,} \end{cases}$$

and the entire analysis could be completed in the base theory $\Sigma_1^1\text{-AC}$.

We should point out that a lower bound for the proof-theoretic strength of $\text{CM} + (\text{W1}) + (\text{W2})$ still remains an open problem, and I suspect that the addition of extensional assertions alone does not cause a concrete increase in strength from $\Sigma_1^1\text{-AC}$ to ATR . Rather, the fact that ATR is needed here is because only in the base theory ATR_0 would we have sufficiently robust infrastructure to work with the weak pca Hyp . We will see more evidence and discussion for this need of a strong base theory in the next section, where we consider other weak fragments of $\text{CM} + \text{GWO}$.

3.6 Predicative fragments of GWO

Now, as briefly mentioned in section 3.3, Weaver believed that a *predicative* extension should never admit the full transfinite induction scheme (W2) for arbitrary third-order formulae. In his own words in [53],

[transfinite induction] should only be asserted for formulas that do not contain $\prec$, for reasons having to do with the circularity involved in making sense of a relation that is well-ordered with respect to properties that are defined in terms of that relation.

Indeed, our ordinal analysis of full $\text{CM} + \text{GWO}$ suggests that Weaver’s remark aligns well with his philosophical stances, since he is known to criticise many other theories with proof-theoretic strength at the Bachmann–Howard ordinal as impredicative, such as BI , ID_1 or the set theories KP and CZF — the latter two for their unrestricted set induction schemes, precisely (cf. [57]).

However, while the rejection of impredicative constructions is understandable from his philosophical viewpoint, Weaver’s suggestion quoted here fails to resolve the problem. If we read it in the usual sense, i.e. we forbid the primitive relation symbol $\prec$ from occurring in φ but still allow the occurrences of any other free parameters, then given any formula $\varphi(X)$, one can simply apply the transfinite induction scheme to the formula $\tilde{\varphi}(X)$ instead, where every occurrence of $X \prec Y$ has been replaced by $\langle X, Y \rangle \in \mathbf{W}$. Here $\mathbf{W}$ is an unused, free third-order parameter, and $\langle X, Y \rangle$ is a fixed encoding of an ordered pair

of second-order objects as a single second-order object, for example, as

$$\langle X, Y \rangle = \{\langle 0, n \rangle : n \in_1 X\} \cup \{\langle 1, m \rangle : m \in_1 Y\}.$$

By decidable comprehension, we can simply take $\mathbf{W} = \{\langle X, Y \rangle : X \prec Y\}$ later to derive transfinite induction for the original formula φ . Therefore, such superficial constraint on φ does not actually restrict the theory in any way.

If we are to read this proposal in a harsher way, by either forbidding the occurrence of any free (third-order) parameter in φ at all, or maintaining that the axiom scheme of decidable comprehension is postulated for the restricted language of CM only and cannot be applied to formulae containing the new symbol $\prec$, then the utility of the extension will be severely impaired.

For example, as we have explored in section 3.3, the proof of traditional mathematical results like the existence of bases using the global well-ordering axioms will heavily involve recursive constructions along the well-ordering and then applications of the induction axiom to prove properties of such constructions. It is unclear how any of those results, including the general version of the Hahn–Banach theorem that Weaver himself claimed to have proved, can be established without accessing the initial segment, i.e. second-order sets $Y \prec X$ for a given X , in a meaningful way in the induction axiom.

Moving away from Weaver’s original proposal, we would still like to find interesting weak fragments lying between CM and CM + GWO such that both

- (1) it has significantly weaker strength than full CM + GWO; and
- (2) it can still prove most of the results surrounding Zorn’s lemma that we examined in section 3.3.

We shall look at some candidates along this direction.

Observe that for any decidable formula $\varphi(X)$, there is some third-order object $\mathbf{X}$ such that

$$\forall X (\varphi(X) \leftrightarrow X \in_2 \mathbf{X})$$

by decidable comprehension, i.e. every decidable formula is equivalent to an atomic formula of this specific form. The class of decidable formulae is very robust since they are closed under arithmetic quantifiers $\forall n, \exists n$ by numerical omniscience (cf. Lemma 3.1.3), and bounded second-order quantifiers $\forall Y \prec X, \exists Y \prec X$ as well, by Lemma 3.3.1 which only requires the axiom of countable initial segments (W3).

Likewise, we can consider the class of Σ -formulae in the language of CM as the class of formulae $\varphi(X)$ equivalent to some formula of the form $\exists Y \langle X, Y \rangle \in_2 \mathbf{X}$. This class contains all formulae with no third-order quantifiers or second-order universal quantifiers, and only one second-order existential quantifier occurring in strictly positive positions, i.e. not behind a negation or in the antecedent of an implication; the class of Σ -formulae is also closed under first-order and bounded second-order quantifiers, thanks to the recursion scheme (3) allowing us to swap first- and second-order quantifiers.

Therefore, we may consider two single axioms

$$(W2_0) \quad \forall X (\forall Y \prec X Y \in_2 \mathbf{X} \rightarrow X \in_2 \mathbf{X}) \rightarrow \forall X X \in_2 \mathbf{X},$$

$$(W2_\Sigma) \quad \forall X (\forall Y \prec X \exists Z \langle Y, Z \rangle \in_2 \mathbf{X} \rightarrow \exists Z \langle X, Z \rangle \in_2 \mathbf{X}) \rightarrow \forall X \exists Z \langle X, Z \rangle \in_2 \mathbf{X}.$$

Let GWO_0 denote the axioms $(W1) + (W2_0) + (W3)$, where the transfinite induction scheme is replaced by the single axiom $(W2_0)$; and GWO_Σ denote the axioms $(W1) + (W2_\Sigma) + (W3)$ likewise.

I believe that these fragments may be argued as ‘somewhat predicative’ in Weaver’s standards, since by forbidding transfinite induction over a universal second-order quantifier, one essentially prevents construction using information about the entire second-order domain at each step of the inductive process; whereas on the other hand, querying about the well-ordering in the constructed initial segment so far should not be as dangerous.

$\text{CM} + \text{GWO}_\Sigma$, particularly, has good prospects in applications as well. Looking back on our use of the transfinite induction scheme, one should observe that, as remarked in footnote 16, our proof of Zorn’s lemma as Proposition 3.3.2 only uses the axiom once on a decidable formula and again on some $\exists Z \theta(X, Z)$ where $\theta(X, Z)$ is decidable, i.e. a Σ -formula. Thus, Proposition 3.3.2 and all of its consequences should still hold in $\text{CM} + \text{GWO}_\Sigma$. The fragment $\text{CM} + \text{GWO}_0$ though, is weaker and cannot accommodate as much.

As for their proof-theoretic strength, we shall first look at exactly the same interpretation as in section 3.4 and adapt Proposition 3.4.10 to realise $(W2_\Sigma)$. Notice that the key use of bar induction in the proof for this instance is then on a claim of the form

$$ha \downarrow \Vdash (\exists Z \langle X, Z \rangle \in_2 \mathbf{X}) [H^{-1}a/X, f/\mathbf{X}]$$

for all a in a given well-ordering and fixed $h \in \text{Hyp}$, $f \in S_3$. By definition of

the realisability conditions, this is equivalent²⁵ to saying

$$ha\downarrow \wedge \mathbf{p}_0(ha) \in S_2 \wedge f(g(H^{-1}a)(\mathbf{p}_0(ha))) = 1,$$

where $g = \lambda den. \mathbf{d}_{\mathbb{N}}(\mathbf{p}_0 n)0(d(\mathbf{p}_1 n))(\mathbf{d}_{\mathbb{N}}(\mathbf{p}_0 n)1(e(\mathbf{p}_1 n))0)$ such that $gde \in S_2$ computes our ordered pair encoding

$$\langle X, Y \rangle = \{\langle 0, n \rangle : n \in_1 X\} \cup \{\langle 1, m \rangle : m \in_1 Y\}$$

for two second-order objects represented by $d, e \in S_2$ respectively. The conjuncts $ha\downarrow$ and $\mathbf{p}_0(ha) \in S_2$ are both Δ_2^1 , so we only need bar induction for Δ_2^1 -formulae to realise $(W2_{\Sigma})$.

As it turns out, one gets nice base theories to work with if the formula in the bar induction scheme is restricted to Π_n^1 for some $n \in \mathbb{N}$. Using Simpson's notations in [43], these fragments shall be denoted as $\Pi_n^1\text{-TI}_0$, whereas $\Pi_n^1\text{-TI}$ denote their respective extensions with unrestricted induction on the natural numbers. We can conclude that:

Proposition 3.6.1. $\Pi_2^1\text{-TI}$ suffices to realise $(W2_{\Sigma})$, and thus all of $\text{CM} + \text{GWO}_{\Sigma}$.

The theories $\Pi_n^1\text{-TI}$ are studied in [39] where it is concluded that $|\Pi_n^1\text{-TI}| = \vartheta_{\Omega^{\varepsilon_0}(n-1)}(0)$ for $n \geq 2$, where $\Omega^{\alpha}(k)$ denotes the power tower

$$\Omega^{\alpha}(0) = \alpha, \quad \Omega^{\alpha}(k+1) = \Omega^{\Omega^{\alpha}(k)}.$$

It follows that the proof-theoretic strength of $\text{CM} + \text{GWO}_{\Sigma}$ is bounded above by $\vartheta_{\Omega^{\varepsilon_0}}(0)$.

On the other hand, $\text{CM} + \text{GWO}_0$, while less capable, will require a weaker base theory to realise. This requires a different argument than Proposition 3.4.10:

Proposition 3.6.2. ATR suffices to realise $(W2_0)$, and thus all of $\text{CM} + \text{GWO}_0$.

Proof. Fix some $h \in S_3$ and suppose that one has a realiser for the antecedent

²⁵Pedantically, one reads $\langle X, Z \rangle \in_2 \mathbf{X}$ as an abbreviation for $\exists W \in_2 \mathbf{X} W = \langle X, Z \rangle$ where the inner formula is additionally the defining formula of second-order ordered pairs, and thus it should have a realiser with some internal structure. However, such a realiser is easily computable in Hyp given correct assignments to the variables X, Z and W . Thus it is not hard to modify the proof of Proposition 3.4.10 in this specific case so that we just work with a function h where the second component of every $ha\downarrow$ is trivial, as if $\langle X, Z \rangle \in_2 \mathbf{X}$ is an atomic statement.

$\forall X (\forall Y \prec XY \in_2 h \rightarrow X \in_2 h)$. This implies

$$\forall x \in S_2 (\forall y \in S_2 (Hy \prec Hx \rightarrow hy = 1) \rightarrow hx = 1),$$

and it suffices to show that $\forall x \in S_2 hx = 1$ as then any $d \in \text{Hyp}$ that is total on the appropriate inputs, e.g. $d = \lambda hf.0$, would realise the axiom as desired.

Suppose that this is not the case, then there exists some $x \in S_2$ such that $hx = 0$, and it follows that we have some $y \in S_2$ such that $Hy \prec Hx$, where $h(H^{-1}(Hy)) = hy = 0$ because $h \in S_3$ respects extensional equivalences, i.e.

$$W = \{a \in \mathcal{H} : a \prec Hx \wedge a = H(H^{-1}a) \wedge h(H^{-1}a) = 0\}$$

is nonempty.

Now, just as in the proof of Proposition 3.4.10, we argue using the totality of $C^*(Hx)$ that W is a Δ_1^1 -subclass of $\mathbb{N}$ and thus a set in ATR by Δ_1^1 -comprehension. Since $\prec$ is a well-ordering on $\mathcal{H}$, this set W must contain a $\prec$ -minimal element b . However, there must then be some $y \in S_2$ such that $hy = 0$ and

$$Hy \prec H(H^{-1}b) = b \prec Hx,$$

contradicting the minimality of b . Thus, we must have $\forall x \in S_2 hx = 1$ by contradiction. $\square$

It follows that the proof-theoretic strength of $\text{CM} + \text{GWO}_0$ is bounded above by the ordinal Γ_{ε_0} .

It is worth remarking briefly that while these ordinals lie beyond the well-known traditional ‘limit of predicativity’, Γ_0 as worked out by Feferman [14] and Schütte [41], Weaver’s mathematical conceptualist would not have difficulties seeing them as predicative. This is because Weaver views any countable iterations as concrete — which is a stronger philosophical assumption than one can infer from Feferman or Schütte’s methodology — and thus, as he made explicit in [56, section 1.4], he agrees with the following criticism in Howard [20]: suppose that $(\gamma_n)_{n \in \mathbb{N}}$ is a sequence of ordinals with limit Γ_0 , and that a predicativist has a uniform way of justifying each γ_n as well-ordered from the well-foundedness of preceding ordinals, then he should immediately see the entire, countable process as a predicatively valid way to justify that ‘for all $n \in \mathbb{N}$, γ_n is well-ordered’, and thus derive that Γ_0 is well-ordered as well.

Assuming that this argument is philosophically sound, a predicativist like

Weaver should easily reach the conclusion that ordinals as large as Γ_{ε_0} are predicatively well-ordered; and following his claims about iterated truth theories in [56, sections 2.8–2.9], he might be able to say the same for $\vartheta_{\Omega^{\varepsilon_0}}(0)$ as well. His methods, however, are unrelated to the focus of this thesis and will not be elaborated here.

On the other hand, I do not presently have lower bound arguments for these theories, thus it remains open whether these upper bounds for the weak fragments of $\text{CM} + \text{GWO}$ are tight; it could well be the case that $\text{CM} + \text{GWO}_{\Sigma}$ is actually weaker than ATR_0 .

Indeed, one could observe that the way we utilise the global well-ordering in $\text{CM} + \text{GWO}_{\Sigma}$ is very similar to how Σ_1 -set induction, i.e. the axiom

$$\forall x (\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x)$$

for any Σ_1 -formula $\varphi(x)$, is used in Kripke–Platek set theory KP and the very same proof of Zorn’s lemma can be replicated in $\text{KP}^- + \Sigma_1$ -set induction + $V = L$, where KP^- denotes the base theory of KP (with infinity) without the axiom of foundation. Since other axioms of CM like dependent choice do not contribute much to the strength of an intuitionistic theory (because they are trivially realised in a realisability interpretation using the recursion theorem and sufficient induction, cf. Lemma 3.2.4), we conjecture that:

Conjecture 3.6.3. *$\text{CM} + \text{GWO}_{\Sigma}$ has a much lower proof-theoretic ordinal than $\vartheta_{\Omega^{\varepsilon_0}}(0)$; it might be equiconsistent with*

$$\text{KP}^- + \Sigma_1\text{-set induction} + \text{full induction on } \omega.$$

It is maintained by Rathjen [29] that the latter theory has the proof-theoretic ordinal²⁶

$$\varphi_{\varphi_{\varepsilon_0}(0)}(0),$$

credited to an obscure paper of Cantini [8]; this is far below even Γ_0 .

However, when we interpret the second-order objects as S_2 in the realisability model presented in this thesis, we need ATR_0 at least to attach a non-trivial well-ordering to this set, as we did in section 3.4. Thus, I suspect that an

²⁶The precise result credited to Cantini in [29] is that $\alpha = \varphi_{\varepsilon_0}(0)$ is the least ordinal such that L_{α} satisfies all Σ_1 -consequences of the given extension of KP^- . It then follows from [29, Theorem 2.1] and a folklore result in [29, Remark 2.2] that the theory has proof-theoretic ordinal $\varphi_{\alpha}(0) = \varphi_{\varphi_{\varepsilon_0}(0)}(0)$.

entirely different interpretation might be needed to generate results toward this conjecture and decide that it shall not be resolved within the scope of this thesis.

Chapter 4

Preliminaries in intuitionistic set theory

For the remainder of this thesis, we will shift our focus to the realisability of first-order intuitionistic set theories, i.e. formal theories in the first-order language that consists of a sole binary relation symbol $\in$. In this chapter, we survey and prove some preliminary results crucial for the developments in all later chapters, including presentations and comparisons of the key fragments of set theory that will be investigated. We will reproduce a standard realisability argument for CZF in CZF itself in Rathjen [31], which will serve as a template for the varied realisability models we shall construct later, and we will examine some basic properties of Gödel’s constructible universe L inside this interpretation.

4.1 Constructive Zermelo–Fraenkel set theory

The main results in the set-theoretic part of this thesis will be about a well-known system called *constructive Zermelo–Fraenkel set theory*, CZF, and its extension by adding the powerset axiom. This pair of first-order theories is sandwiched between *intuitionistic Kripke–Platek set theory* IKP, which is a proper subtheory of CZF, and the much stronger *intuitionistic Zermelo–Fraenkel set theory* IZF. The book draft [3] by Aczel and Rathjen serves as a standard reference for these constructs.

We can begin defining them by letting IKP be the same recursive set of axioms as classical Kripke–Platek set theory KP over intuitionistic logic, with the condition that the set induction scheme must be used instead of the foundation

scheme²⁷. In other words, we have the following axioms:

- (1) the axiom of extensionality,

$$\forall a \forall b (\forall x (x \in a \leftrightarrow x \in b) \leftrightarrow a = b);$$

- (2) the pairing axiom,

$$\forall a \forall b \exists y \forall x (x \in y \leftrightarrow x = a \vee x = b);$$

- (3) the union axiom,

$$\forall a \exists b \forall x (x \in b \leftrightarrow \exists u \in a x \in u);$$

- (4) the axiom of infinity, $\exists a \text{Ind}(a)$, where

$$\begin{aligned} \text{Ind}(a) &: \Longleftrightarrow \exists y \in a \forall z \neg z \in y \wedge \forall x \in a \exists y \in a y = x^+, \\ y = x^+ &: \Longleftrightarrow \forall z (z \in y \leftrightarrow z \in x \vee z = x); \end{aligned}$$

- (5) the scheme of Δ_0 -separation,

$$\forall a \exists y \forall x (x \in y \leftrightarrow x \in a \wedge \varphi(x)),$$

for any Δ_0 -formula $\varphi(x)$, i.e. a formula that may contain bounded quantifiers $\forall y \in z$, $\exists y \in z$ but not unbounded quantifiers;

- (6) the scheme of Δ_0 -collection,

$$\forall x \in a \exists y \theta(x, y) \rightarrow \exists b \forall x \in a \exists y \in b \theta(x, y),$$

for any Δ_0 -formula $\theta(x, y)$;

²⁷The foundation scheme is an analogue of the axiom of foundation in ZF and written as

$$\exists x \varphi(x) \rightarrow \exists x (\varphi(x) \wedge \forall y \in x \neg \varphi(y))$$

for any first-order formula φ , such as in [4, section I.2]. This is classically equivalent to set induction, but intuitionistically stronger, as it implies the law of excluded middle immediately when one applies it to

$$\varphi(x) : \Longleftrightarrow x = 1 \vee (x = 0 \wedge \psi)$$

for an arbitrary formula ψ (cf. [3, Proposition 10.4.1]). On the other hand, any realisability argument, such as the one we will soon see in section 4.2, demonstrates that set induction cannot imply the law of excluded middle.

(7) the *set induction* scheme,

$$\forall a (\forall x \in a \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \varphi(a),$$

for any first-order formula φ .

For some caveats, we treat equality as a primitive logical symbol instead of an abbreviation for extensional equality. We also see bounded quantifiers $\forall y \in z$, $\exists y \in z$ as separate primitive constructs in the language instead of abbreviations for some corresponding formulae with unbounded quantifiers, since it is routine to provide different conditions for bounded and unbounded quantifiers in many realisability interpretations (cf. section 4.2). This means that we will assume as axioms

$$\begin{aligned} \forall x (x \in a \rightarrow \varphi(x)) &\leftrightarrow \forall x \in a \varphi(x), \\ \exists x (x \in a \wedge \varphi(x)) &\leftrightarrow \exists x \in a \varphi(x), \end{aligned}$$

for any formula φ and need to check that they are realised by our realisability conditions. However, we still opt to not include them in axiom listings for particular theories like IKP, since it remains ambiguous whether they are ‘logical’ or ‘non-logical’ axioms of the system.

We shall also remark here that, in the context of weak set theories like KP or IKP, many authors prefer to use the axiom of strong infinity instead of (4), namely

$$\exists a (\text{Ind}(a) \wedge \forall b (\text{Ind}(b) \rightarrow \forall x \in a x \in b))$$

(cf. [3, Chapter 3]), since in order to define the least inductive set ω as the intersection of all inductive sets, one seemingly needs to invoke an unavailable instance of Π_1 -separation. However, this is in fact not necessary, as it follows from the following lemma that the usual axiom of infinity (4) already implies the axiom of strong infinity:

Lemma 4.1.1 (IKP). *There is a Δ_0 -formula $\varphi(x)$ such that for any set a satisfying $\exists b \subseteq a \text{Ind}(b)$, the set*

$$\omega = \{x \in a : \varphi(x)\}$$

is precisely the least inductive set in the universe, i.e. we have $\text{Ind}(\omega)$ and $\omega \subseteq b$ for any b satisfying $\text{Ind}(b)$. It follows immediately from the axiom of extensionality that such ω is unique and does not depend on the choice of a .

Proof. We consider the formula²⁸

$$\varphi(x) \quad :\Longleftrightarrow \quad (x = \emptyset \vee \exists y \in x \, x = y^+) \wedge \forall y \in x \, (y = \emptyset \vee \exists z \in x \, y = z^+),$$

where $x = \emptyset$ is the Δ_0 -formula $\forall y \in x \, \neg y \in x$ denoting that x is empty.

Assume that $\varphi(x)$ holds for some set x . We first show by set induction that every $y \in x$ is transitive, i.e. $\forall z \in y \, \forall w \in z \, w \in y$. This is because each $y \in x$ must be either the empty set or a successor z^+ where $z \in x$. In the latter case, $z \in y$ so the inductive hypothesis implies that z is transitive, which immediately entails that $y = z^+$ is also transitive. Since x itself is either the empty set or a successor y^+ where $y \in x$, we know x is transitive as well.

For any $y \in x$ and $z \in y$, we then have $z \in x$ and thus $z = \emptyset \vee \exists w \in x \, z = w^+$ where the latter disjunct implies $\exists w \in y \, z = w^+$ by transitivity of y . It follows that

$$\forall x \, (\varphi(x) \rightarrow \forall y \in x \, \varphi(y)).$$

Fix any set b satisfying $\text{Ind}(b)$, it then follows easily from set induction that $\forall x \, (\varphi(x) \rightarrow x \in b)$, and thus $\omega = \{x \in a : \varphi(x)\} \subseteq b$ as desired. It now remains to show that $\text{Ind}(\omega)$ holds.

Fix some $b \subseteq a$ satisfying $\text{Ind}(b)$, and let $\emptyset$ denote the necessarily unique empty set in b , then clearly $\emptyset \in \omega$. For any $x \in \omega$, we have $\varphi(x)$ and thus $x \in b$ as shown above. It follows that its successor $x^+ = x \cup \{x\}$ lies in b and thus in a . Now, it is easy to check that $\varphi(x)$ implies $\varphi(x^+)$ and thus $x^+ \in \omega$ by definition. $\square$

We then have the following new axiom schemes in CZF:

(6') the *strong collection* scheme,

$$\forall x \in a \, \exists y \, \theta(x, y) \rightarrow \exists b \, (\forall x \in a \, \exists y \in b \, \theta(x, y) \wedge \forall y \in b \, \exists x \in a \, \theta(x, y)),$$

for any first-order formula θ ;

(8) the *subset collection* scheme,

$$\begin{aligned} & \exists c \, \forall u \, (\forall x \in a \, \exists y \in b \, \psi(x, y, u) \\ & \rightarrow \exists d \in c \, (\forall x \in a \, \exists y \in d \, \psi(x, y, u) \wedge \forall y \in d \, \exists x \in a \, \psi(x, y, u))), \end{aligned}$$

²⁸This construction is a simplified version of [50, Lemma 3.7] in my arXiv manuscript.

for any first-order formula ψ ;

and we simply define $\text{CZF} = \text{IKP} + (6') + (8)$. The strong collection scheme (6') entails (6) and is necessary here instead of just the full collection scheme, since in intuitionistic logic, we cannot derive separation instances from collection and thus lack the sufficient amount of separation in CZF to prove strong collection from collection alone.

Subset collection is a more technical assertion that is often mentioned in relation to two more commonly seen axioms:

(8E) the axiom of *exponentiation*, Exp ,

$$\forall a \forall b \exists c \forall f (f \in c \leftrightarrow (f : a \rightarrow b)),$$

where $f : a \rightarrow b$ is the usual Δ_0 -formula expressing that f is a set-theoretic function with domain a and codomain b , essentially that f consists of elements from the Cartesian product $a \times b$, such that every $x \in a$ is paired with exactly one $y \in b$;

(8P) the *powerset* axiom,

$$\forall a \exists y \forall x (x \in y \leftrightarrow x \subseteq a).$$

Following [3, Chapter 5], we know that:

Proposition 4.1.2. (i) IKP proves that (8) implies (8E);

(ii) $\text{IKP} + \text{strong collection (6')}$ proves that (8P) implies (8).

Proof. For (i), fix sets a and b and observe that the Cartesian product $a \times b$ exists by two applications of Δ_0 -collection. For any set-theoretic function $f : a \rightarrow b$, we must have

$$\forall x \in a \exists y \in a \times b (y \in f \wedge \exists z \in b y = \langle x, z \rangle),$$

where $\langle x, z \rangle = \{\{x\}, \{x, z\}\}$ denotes the usual Kuratowski ordered pair construction that populates our Cartesian product. By subset collection, there is some set c (depending on a and b) such that for any $f : a \rightarrow b$, there is $d \in c$ such that

$$\forall x \in a \exists y \in d (y \in f \wedge \exists z \in b y = \langle x, z \rangle) \wedge \forall y \in d y \in f,$$

i.e. $d = f$, by virtue of f being a function. It follows that

$$\{f \in c : (f : a \rightarrow b)\},$$

which is a set by Δ_0 -separation, consists precisely of all functions from a to b .

For (ii), we simply take $c = \mathcal{P}(b)$, the powerset of b , and consider any u such that $\forall x \in a \exists y \in b \psi(x, y, u)$ holds. By strong collection, there is some set d such that

$$\forall x \in a \exists y \in d \psi(x, y, u) \wedge \forall y \in d \exists x \in a (y \in b \wedge \psi(x, y, u)).$$

Note that this ensures $d \subseteq b$, thus $d \in c = \mathcal{P}(b)$ as desired. $\square$

In classical set theory, we can trivially construct some powerset $\mathcal{P}(x)$ from the set of functions $f : x \rightarrow \{0, 1\}$ as

$$\mathcal{P}(x) = \{\{y \in x : \langle y, 1 \rangle \in f\} : (f : x \rightarrow \{0, 1\})\},$$

by Δ_0 -collection, hence (8E) implies (8P) and all three axioms are equivalent. However, this uses the instance of excluded middle $\forall x \forall y (y \in x \vee \neg y \in x)$ and does not hold in CZF.

Notably, Peter Aczel in [2] interpreted CZF in a version of Martin-Löf type theory with a single W -type, which can be easily implemented in any of the theories ID_1 , BI or KP through inductive definitions. More detail about this interpretation will be given in section 5.3, where we adapt the construction to build exotic intuitionistic ordinals within the model. For now, observe that IKP already supports Σ -inductive definitions (see section 4.2) and, in a way analogous to Theorem 3.4.12, contains the intuitionistic ID_1^i as a subtheory. Thus, using again the result in [7] that $ID_1^i \equiv_{\text{Con}} \text{BI}$, we know that CZF has exactly the same proof-theoretic strength as KP at the Bachmann–Howard ordinal.

On the other hand, we denote $\text{CZF}(\mathcal{P}) = \text{CZF} + (8P)$ to be the result of replacing its subset collection scheme by the powerset axiom. Rathjen in [32] showed that $\text{CZF}(\mathcal{P})$ is equiconsistent with the theories $\text{KP}(\mathcal{P})$ and $\text{IKP}(\mathcal{P})$, i.e. classical and intuitionistic Kripke–Platek set theory with a new primitive unary function symbol²⁹ $\mathcal{P}$ defined to be the powerset operation. The inter-

²⁹ As stipulated in this manner, the primitive operation $\mathcal{P}$ will be allowed to occur in Δ_0 -formulae used in the separation and collection axioms, which makes these systems different from simply adjoining the powerset axiom (8P) to KP or IKP. The latter are weaker since

pretation of $\text{CZF}(\mathcal{P})$ in $\text{KP}(\mathcal{P})$ involves a modification to the usual realisability argument known as *weak realisability*, which will become a key ingredient in section 6.2 later.

We can immediately see that $\text{KP}(\mathcal{P})$ proves the consistency of full second-order arithmetic Z_2 , i.e. the base theory ACA_0 together with unrestricted comprehension axiom for all second-order formulae. BI is a proper subtheory of Z_2 and thus correspondingly, CZF with only the subset collection axiom is much weaker than the variant $\text{CZF}(\mathcal{P})$ with the powerset axiom.

Before concluding this section, we also note that:

Theorem 4.1.3 ([3, Corollary 4.2.8]). *CZF and $\text{CZF}(\mathcal{P})$ are both equivalent to ZF over classical logic.*

Proof. By Proposition 4.1.2, CZF implies the exponentiation axiom Exp . Observe that classically, strong collection implies separation, set induction implies foundation and exponentiation implies powerset. Thus we have all the axioms of ZF . $\square$

While this is the case, there exist stronger intuitionistic fragments of ZF . We can let IZF denote $\text{CZF}(\mathcal{P}) + \text{full separation}$, so that IZF consists of essentially the same axioms as ZF except the axiom of foundation is replaced by set induction.

By forcing over a complete Boolean algebra in IZF , Robin Grayson in [18, section 2.3] verified that IZF is equiconsistent with classical ZF . The strength of ZF , however, is enormous even in comparison to $\text{CZF}(\mathcal{P})$ and $\text{KP}(\mathcal{P})$. Rathjen showed as [36, Theorem 4.6] that $\text{KP}(\mathcal{P}) + V = L$ suffices³⁰ to prove $\text{Con}(\text{KP}(\mathcal{P}))$, and $\text{KP}(\mathcal{P}) + V = L$ is an obvious subtheory of $\text{ZF} + V = L$, which is equiconsistent with ZF through the constructible universe construction.

In this thesis, we will mainly be concerned with CZF and $\text{CZF}(\mathcal{P})$; we will not work with models of IZF as it is too strong for many of the questions we analyse here. For features of realisability for IZF , we refer the readers to the PhD thesis of Charles McCarty [26].

powersets in general will not be definable inside a Δ_0 -formula; see [32, section 2.3] for more discussion.

³⁰This is a slightly surprising result, which invokes the recent framework of relativised ordinal analysis developed in [35]. While the proof is not directly related to the topics of this thesis, this will be revisited in section 5.2 in greater detail to provide background for our subsequent investigations.

4.2 A standard realisability model for CZF

We now outline the standard realisability argument for CZF inside CZF in [31], i.e. we show in the given interpretation that for any axiom φ of CZF, CZF proves that φ is realised.

Given any pca $\mathcal{A}$ over natural numbers — such that both $\mathcal{A}$ and its application function are sets in our base theory — we first define the domain $V(\mathcal{A})$ of the interpretation, which we shall borrow the tradition from forcing and call the class of *names*. This will be an uninterpreted realisability model (cf. section 2.4), as with many other realisability interpretations of set theory, and the names will form a very large class, which we construct using Σ -inductive definitions in IKP (cf. [3, section 19.3]).

Σ -inductive definitions

Theorem 4.2.1 (IKP, [3, Theorem 19.3.1]). *Let Φ be a Σ_1 -class of ordered pairs. Then there is a Σ_1 -class $I(\Phi)$ satisfying*

- (i) *for any $\langle x, y \rangle \in \Phi$, if $y \subseteq I(\Phi)$, then $x \in I(\Phi)$;*
- (ii) *for any formula $\varphi(x)$, if*

$$\forall \langle x, y \rangle \in \Phi (\forall z \in y \varphi(z) \rightarrow \varphi(x)),$$

then $\forall x \in I(\Phi) \varphi(x)$.

Proof. We consider the formula

$$\begin{aligned} \psi(u, x, f) \quad : \iff \quad & f \text{ is a function on domain } u^+ = u \cup \{u\} \wedge x \in f(u) \\ & \wedge \forall v \in u^+ \forall y \in f(v) \exists z (\langle y, z \rangle \in \Phi \wedge \exists w \in v z \subseteq f(w)), \end{aligned}$$

which is Σ_1 because we can move the bounded quantifiers in the last conjunct inside unbounded existential quantifiers by Δ_0 -collection. Let

$$I(\Phi) = \{x : \exists u \exists f \psi(u, x, f)\}.$$

To verify (i), fix any $\langle x, y \rangle \in \Phi$ such that $y \subseteq I(\Phi)$. By strong Σ_1 -collection (which is a theorem in IKP, cf. [3, Theorem 19.1.5]), let s be a set such that

$$\forall z \in y \exists \langle u, f \rangle \in s \psi(u, z, f) \wedge \forall \langle u, f \rangle \in s \exists z \in y \psi(u, z, f).$$

Take $q = \bigcup \{u : \langle u, f \rangle \in s\}$ and consider

$$g = \{\langle q^+, \{x\} \rangle, \langle q, y \rangle\} \cup \left\{ \left\langle v, \bigcup_{\langle u, f \rangle \in s, v \in u} f(v) \right\rangle : v \in q \right\}.$$

Clearly, g is a function since $q^+, q \notin q$ by set induction. We also see that for each $\langle u, f \rangle \in s$, $v \in u$, $f(v) \subseteq g(v)$, thus $\psi(q, z, g \upharpoonright q^+)$ for each $z \in y$ and therefore $\psi(q^+, x, g)$. It follows that $x \in I(\Phi)$.

For (ii), fix any $x \in I(\Phi)$ and corresponding u, f such that $\psi(u, x, f)$ holds. Using stipulations in ψ we immediately show by set induction on v that

$$\forall v \in u^+ \forall y \in f(v) \varphi(y),$$

and it follows from $x \in f(u)$ that $\varphi(x)$ holds as well. $\square$

The use of strong Σ_1 -collection here is covered by the strong collection axiom when we work in CZF, thus in CZF we can even generate inductive definitions from any definable class Φ (cf. [3, section 12.1]).

Corollary 4.2.2 (IKP). *Given the generated inductive definition $I(\Phi)$, for any $x \in I(\Phi)$, there is some $y \subseteq I(\Phi)$ such that $\langle x, y \rangle \in \Phi$.*

Proof. Simply take the formula

$$\varphi(x) \quad :\Longleftrightarrow \quad \exists y \subseteq I(\Phi) \langle x, y \rangle \in \Phi$$

and observe that $\varphi(x)$ implies $x \in I(\Phi)$ by condition (i) in Theorem 4.2.1. Thus for any $\langle x, y \rangle \in \Phi$, if $\forall z \in y \varphi(z)$, then $y \subseteq I(\Phi)$, i.e. $\varphi(x)$ holds. Therefore, the statement follows from condition (ii) in Theorem 4.2.1. $\square$

Now, we construct the class of names for our realisability model:

Corollary 4.2.3 (IKP). *There is a Σ_1 -class $V(\mathcal{A})$ satisfying*

$$\forall x (x \in V(\mathcal{A}) \leftrightarrow x \subseteq \mathcal{A} \times V(\mathcal{A})).$$

Proof. Simply let class $\Phi = \{\langle x, y \rangle : x \subseteq \mathcal{A} \times y\}$ and set $V(\mathcal{A}) = I(\Phi)$. The forward direction is trivially ensured by Corollary 4.2.2 whereas for the backward direction, fix $x \subseteq \mathcal{A} \times V(\mathcal{A})$ and take $y = \{z : \langle a, z \rangle \in x\}$ by Δ_0 -collection. Then $\langle x, y \rangle \in \Phi$ and condition (i) in Theorem 4.2.1 ensures that $x \in V(\mathcal{A})$. $\square$

Realisability for CZF

We proceed to define the atomic realisability conditions as class functions $F_{\in}, G_{=} : V(\mathcal{A})^2 \rightarrow \mathcal{P}(\mathcal{A})$ by simultaneous Σ -recursion³¹ in IKP as follows:

$$\begin{aligned} F_{\in}(u, v) &= \{e \in \mathcal{A} : \exists \langle f, x \rangle \in v (\mathbf{p}_0 e \downarrow = f \wedge \mathbf{p}_1 e \downarrow \in G_{=}(u, x))\}, \\ G_{=}(u, v) &= \{e \in \mathcal{A} : \forall \langle f, x \rangle \in u \mathbf{p}_0 e f \downarrow \in F_{\in}(x, v) \\ &\quad \wedge \forall \langle f, x \rangle \in v \mathbf{p}_1 e f \downarrow \in F_{\in}(x, u)\}. \end{aligned}$$

The atomic clauses of the realisability conditions are then

$$\begin{aligned} d \Vdash u \in v &\quad :\Longleftrightarrow \quad d \in F_{\in}(u, v), \\ d \Vdash u = v &\quad :\Longleftrightarrow \quad d \in G_{=}(u, v). \end{aligned}$$

Additionally, as previously mentioned, we will specify realisability for bounded quantifiers as separate clauses given by

$$\begin{aligned} d \Vdash \forall x \in u \varphi &\quad :\Longleftrightarrow \quad \forall \langle f, v \rangle \in u d f \downarrow \Vdash \varphi[v/x], \\ d \Vdash \exists x \in u \varphi &\quad :\Longleftrightarrow \quad \exists \langle f, v \rangle \in u (\mathbf{p}_0 d \downarrow = f \wedge \mathbf{p}_1 d \downarrow \Vdash \varphi[v/x]). \end{aligned}$$

Observe that the forms of realisers for bounded quantifiers are chosen to match the definitions of $F_{\in}$ and $G_{=}$ precisely, so that we immediately have

$$\begin{aligned} (d \Vdash u \in v) &\leftrightarrow (d \Vdash \exists x \in v u = x), \\ (d \Vdash u = v) &\leftrightarrow (d \Vdash u \subseteq v \wedge v \subseteq u), \end{aligned}$$

where $u \subseteq v$ is read as an abbreviation for $\forall x \in u x \in v$.

We will treat $V(\mathcal{A})$ under the realisability conditions as a ‘model’ analogous to classical models in first-order model theory, and write $V(\mathcal{A}) \models \varphi[\vec{u}/\vec{x}]$ to

³¹A pedantic formulation of this can be quite complicated. Observe that an ‘element name’ of some $u \in V(\mathcal{A})$ is not a direct element of u , but rather some

$$x \in \{f, x\} \in \langle f, x \rangle \in u$$

by the Kuratowski pair construction, thus the induction needs to be on the transitive closure $\text{TC}(\{u\})$. Additionally, we perform the recursion on two parameters u and v , while also swapping their positions from time to time (see the second conjunct in the definition of $G_{=}$), so the full formal presentation must be

$$\begin{aligned} &\forall u \forall x \in \text{TC}(\{u\}) \cap V(\mathcal{A}) \forall v \forall y \in \text{TC}(\{v\}) \cap V(\mathcal{A}) \\ &\quad \text{all four constructs } F_{\in}(x, y), F_{\in}(y, x), G_{=}(x, y), G_{=}(y, x) \text{ are well-defined and unique,} \end{aligned}$$

which is then proved by set induction on u first and then on v for each inductive step.

abbreviate for

$$\exists d \in \mathcal{A} d \Vdash \varphi[\vec{u}/\vec{x}]$$

where $\vec{u} \in V(\mathcal{A})^k$.

Now, we first check that the equality axioms for our primitive equality symbol (cf. section 2.3) are realised. Observe that there are precisely two cases for the atomic substitution scheme (E4) (up to renaming of variables):

Lemma 4.2.4 ([31, Lemma 4.2]). *We have fixed $\iota_r, \iota_s, \iota_t, \iota_{s1}, \iota_{s2} \in \mathcal{A}$ such that for any $x, y, z \in V(\mathcal{A})$,*

- (i) $\iota_r \Vdash x = x$;
- (ii) $\iota_s \Vdash x = y \rightarrow y = x$;
- (iii) $\iota_t \Vdash x = y \rightarrow (y = z \rightarrow x = z)$;
- (iv) $\iota_{s1} \Vdash x = y \rightarrow (x \in z \rightarrow y \in z)$;
- (v) $\iota_{s2} \Vdash x = y \rightarrow (z \in x \rightarrow z \in y)$.

Thus the equality axioms (E1)–(E4) are realised.

Proof. (i) We let

$$h = \mathbf{r}(\lambda f g. \mathbf{p}f(\mathbf{p}g g))$$

by recursion, and simply define $\iota_r = \mathbf{p}hh$. By set induction³² on $V(\mathcal{A})$, it is not hard to check that $h \Vdash x \subseteq x$ and $\iota_r \Vdash x = x$ for any $x \in V(\mathcal{A})$.

(ii) $\iota_s = \lambda d. \mathbf{p}(\mathbf{p}_1 d)(\mathbf{p}_0 d)$ is trivial.

(iii) We examine some functions

$$\begin{aligned} s_2 &= \lambda g d e. g(\mathbf{p}_1 e)(d(\mathbf{p}_0 e)), \\ t &= \lambda g d e f. s_2 g e(df), \\ p &= \lambda g d e. \mathbf{p}(tg(\mathbf{p}_0 d)(\mathbf{p}_0 e))(tg(\mathbf{p}_1 e)(\mathbf{p}_1 d)), \\ s_1 &= \lambda g d e. \mathbf{p}(\mathbf{p}_0 e)(pgd(\mathbf{p}_1 e)), \end{aligned}$$

³²Again since names in $V(\mathcal{A})$ are not direct elements of each other, this induction must be formalised on the transitive closure, i.e. proving the formula

$$\forall x \forall u \in \text{TC}(\{x\}) \cap V(\mathcal{A}) (h \Vdash u \subseteq u \wedge \iota_r \Vdash u = u).$$

Same applies to subsequent induction arguments.

and observe that, $\text{fix}^{33} y \in V(\mathcal{A})$ and given some $g \in \mathcal{A}$ such that for any $\langle f, v \rangle \in y$ and any $u, w \in V(\mathcal{A})$,

$$g \Vdash u = v \rightarrow (v \in w \rightarrow u \in w),$$

then we have, for any $x, z \in V(\mathcal{A})$,

$$s_2 g \Vdash y \subseteq z \rightarrow (x \in y \rightarrow x \in z),$$

$$tg \Vdash x \subseteq y \rightarrow (y \subseteq z \rightarrow x \subseteq z),$$

$$pg \Vdash x = y \rightarrow (y = z \rightarrow x = z),$$

$$s_1 g \Vdash x = y \rightarrow (y \in z \rightarrow x \in z).$$

Thus, by recursion, if we take $g = \lambda de. \mathbf{r}(\lambda ah. s_1 h(\mathbf{p}_0 a)(\mathbf{p}_1 a))(\mathbf{p} de)$, then we can see that $g, s_2 g, tg, pg, s_1 g$ realise their corresponding formulae for all $x, y, z \in V(\mathcal{A})$ by set induction on y . It suffices now to take $\iota_t = pg$.

(iv) Simply use s_1 from the induction above as $\iota_{s1} = s_1 g$.

(v) Simply use s_2 from the induction above as $\iota_{s2} = \lambda d. s_2 g(\mathbf{p}_0 d)$. $\square$

Through a standard (metatheoretic) induction on the complexity of formula φ , we know that the equality axioms prove

$$x = y \rightarrow (\varphi(x) \rightarrow \varphi(y))$$

for any formula φ . By Proposition 2.3.1, it follows that there is some $e \in \mathcal{A}$, depending on φ , such that

$$e \Vdash x = y \rightarrow (\varphi(x) \rightarrow \varphi(y)).$$

We use this to show that the axioms for the bounded quantifiers are realised:

Lemma 4.2.5. *The following axioms are realised:*

$$\forall x (x \in a \rightarrow \varphi(x)) \leftrightarrow \forall x \in a \varphi(x),$$

$$\exists x (x \in a \wedge \varphi(x)) \leftrightarrow \exists x \in a \varphi(x).$$

Proof. First see that $\lambda df. d(\mathbf{p} f \iota_r)$ realises $\forall x (x \in a \rightarrow \varphi(x)) \rightarrow \forall x \in a \varphi(x)$, whereas $\lambda d. \mathbf{p}(\mathbf{p}(\mathbf{p}_0 d) \iota_r)(\mathbf{p}_1 d)$ realises $\exists x \in a \varphi(x) \rightarrow \exists x (x \in a \wedge \varphi(x))$.

³³It is important that we focus on the variable y through this induction, since when p invokes t above, it will swap the variables x and z , but y remains unchanged and provides us with the appropriate inductive hypothesis to work with.

For the converses, fix $e \in \mathcal{A}$ such that $e \Vdash x = y \rightarrow (\varphi(x) \rightarrow \varphi(y))$. Then $\lambda df. e(\iota_s(\mathbf{p}_1 f))(d(\mathbf{p}_0 f))$ realises $\forall x \in a \varphi(x) \rightarrow \forall x (x \in a \rightarrow \varphi(x))$, whereas $\lambda d. \mathbf{p}(\mathbf{p}_{00} d)(e(\mathbf{p}_{01} d)(\mathbf{p}_1 d))$ realises $\exists x (x \in a \wedge \varphi(x)) \rightarrow \exists x \in a \varphi(x)$. $\square$

Moving on to the axioms of CZF, where [31, section 5] serves as a primary reference, we have:

Lemma 4.2.6. *The axiom of extensionality (1) is realised:*

$$\forall a \forall b (\forall x (x \in a \leftrightarrow x \in b) \leftrightarrow a = b).$$

Proof. It suffices to realise $\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b$ since the converse is a consequence of the equality axioms. We use the fact that $(d \Vdash a = b) \leftrightarrow (d \Vdash a \subseteq b \wedge b \subseteq a)$, and it follows that

$$\lambda d. \mathbf{p}(\lambda f. \mathbf{p}_0 d(\mathbf{p} f \iota_r))(\lambda f. \mathbf{p}_1 d(\mathbf{p} f \iota_r)) \Vdash \forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b. \quad \square$$

For the pairing axiom (2) and union axiom (3), we define

$$\begin{aligned} \{a, b\}^\bullet &= \{\langle 0, a \rangle\} \cup \{\langle 1, b \rangle\}, \\ \bigcup^\bullet a &= \{\langle \mathbf{p} f g, y \rangle : \langle f, x \rangle \in a, \langle g, y \rangle \in x\} \end{aligned}$$

for any $a, b \in V(\mathcal{A})$. For pairing, observe that simply

$$\mathbf{p}(\lambda d. d)(\lambda d. d) \Vdash x \in \{a, b\}^\bullet \leftrightarrow x = a \vee x = b,$$

whereas for union, note that our definition ensures

$$(\mathbf{p}(\mathbf{p} f g) d \Vdash x \in \bigcup^\bullet a) \leftrightarrow (\mathbf{p} f(\mathbf{p} g d) \Vdash \exists u \in a x \in u),$$

and hence we have

$$\begin{aligned} \lambda d. \mathbf{p}(\mathbf{p}_{00} d)(\mathbf{p}(\mathbf{p}_{01} d)(\mathbf{p}_1 d)) \Vdash x \in \bigcup^\bullet a \rightarrow \exists u \in a x \in u, \\ \lambda d. \mathbf{p}(\mathbf{p}(\mathbf{p}_0 d)(\mathbf{p}_{10} d))(\mathbf{p}_{11} d) \Vdash \exists u \in a x \in u \rightarrow x \in \bigcup^\bullet a. \end{aligned}$$

For the axiom of infinity (4), we recursively define

$$\begin{aligned} \bar{n} &= \{\langle m, \bar{m} \rangle : m \in n\} \in V(\mathcal{A}) \quad \text{for all } n \in \omega, \\ \bar{\omega} &= \{\langle n, \bar{n} \rangle : n \in \omega\} \in V(\mathcal{A}). \end{aligned}$$

We see that $\bar{\omega}$ is realised as the least inductive set precisely:

Lemma 4.2.7. *Both $\text{Ind}(\bar{\omega})$ and $\text{Ind}(a) \rightarrow \bar{\omega} \subseteq a$ are realised. Note that the former suffices to show that $\exists a \text{Ind}(a)$ is realised.*

Proof. Let

$$g = \lambda n. \mathbf{p}(\lambda d. \mathbf{d}_{\mathbb{N}} d n (\mathbf{p}1(\mathbf{p}_1 d)) (\mathbf{p}0 d)) (\lambda d. \mathbf{d}_{\mathbb{N}} (\mathbf{p}_0 d) 0 d (\mathbf{p}n d))$$

and verify that for any $n \in \omega$, $gn \Vdash \overline{n+1} = \bar{n}^+$. It follows that

$$\mathbf{p}(\mathbf{p}00)(\lambda n. \mathbf{p}(\mathbf{s}_{\mathbb{N}} n)(gn)) \Vdash \text{Ind}(\bar{\omega}).$$

For the latter, we instead check that

$$\forall x \in \bar{\omega} \left((x = \emptyset \vee \exists y \in x \ x = y^+) \wedge \forall y \in x \ (y = \emptyset \vee \exists z \in x \ y = z^+) \right)$$

is realised. This is clearly the case as³⁴ $\mathbf{d}_{\mathbb{N}}^* n 0 (\mathbf{p}0(\lambda a. 0)) (\mathbf{p}1(\mathbf{p}(\mathbf{p}_{\mathbb{N}} n)(g(\mathbf{p}_{\mathbb{N}} n))))$ realises $\bar{n} = \emptyset \vee \exists y \in \bar{m} \ \bar{n} = y^+$ for any $m \geq n \in \omega$. Thus, once we show that all axioms of CZF are realised, it follows from Lemma 4.1.1 inside the realisability model that $\bar{\omega}$ is precisely the least inductive set as we constructed in that lemma. $\square$

We will leave the scheme of Δ_0 -separation (5) to the end to apply a trick that depends on collection. So we first see that:

Lemma 4.2.8. *The set induction scheme (7) is realised:*

$$\forall a \ (\forall x \in a \ \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \ \varphi(a).$$

Proof. We claim that for any $e \in \mathcal{A}$, $\lambda f. \mathbf{r}(\lambda eg. fg)e$ realises the axiom. Suppose that $f \Vdash \forall a (\forall x \in a \ \varphi(x) \rightarrow \varphi(a))$, then we invoke set induction on $x \in V(\mathcal{A})$ and, from the inductive hypothesis that for any $e \in \mathcal{A}$ and $\langle d, y \rangle \in x$,

$$\mathbf{r}(\lambda eg. fg)e \Vdash \varphi(y),$$

we derive that $f(\lambda e. \mathbf{r}(\lambda eg. fg)e) \Vdash \varphi(x)$. Thus the claim follows.

We can simply take $\lambda f. \mathbf{r}(\lambda eg. fg)\mathbf{k}$, for example, for a concrete realiser. $\square$

³⁴One caveat is that $x = \emptyset$ as defined in Lemma 4.1.1 is the Δ_0 -formula $\forall y \in x \ \neg y \in x$ instead of $\forall y \neg y \in x$, so it demands a slightly more complicated realiser.

Lemma 4.2.9. *The strong collection scheme (6') is realised:*

$$\forall x \in a \exists y \theta(x, y) \rightarrow \exists b (\forall x \in a \exists y \in b \theta(x, y) \wedge \forall y \in b \exists x \in a \theta(x, y)).$$

Proof. For $a \in V(\mathcal{A})$, suppose that $f \Vdash \forall x \in a \exists y \theta(x, y)$, i.e. for any $\langle e, x \rangle \in a$, $fe \downarrow \Vdash \theta(x, y)$ for some $y \in V(\mathcal{A})$. By strong collection in the base theory, there is some set s such that

- (i) for any $\langle e, x \rangle \in a$, there is $\langle e, y \rangle \in s$ such that $fe \Vdash \theta(x, y)$, and
- (ii) for any $u \in s$, we have $u = \langle e, y \rangle \in \mathcal{A} \times V(\mathcal{A})$ and there is $\langle e, x \rangle \in a$ such that $fe \Vdash \theta(x, y)$.

Note that condition (ii) ensures that $s \in V(\mathcal{A})$. We see that $\lambda e. \mathbf{pe}(fe)$ realises both $\forall x \in a \exists y \in s \theta(x, y)$ and $\forall y \in s \exists x \in a \theta(x, y)$. It follows that $\lambda f. \mathbf{p}(\lambda e. \mathbf{pe}(fe))(\lambda e. \mathbf{pe}(fe))$ realises the axiom. $\square$

Lemma 4.2.10. *The subset collection scheme (8) is realised:*

$$\begin{aligned} & \exists c \forall u (\forall x \in a \exists y \in b \psi(x, y, u) \\ & \rightarrow \exists d \in c (\forall x \in a \exists y \in d \psi(x, y, u) \wedge \forall y \in d \exists x \in a \psi(x, y, u))), \end{aligned}$$

Proof. Fix $a, b, u \in V(\mathcal{A})$ and suppose that $f \Vdash \forall x \in a \exists y \in b \psi(x, y, u)$, i.e. for any $\langle e, x \rangle \in a$, there exists $\langle d, y \rangle \in b$ such that $\mathbf{p}_0(fe) \downarrow = d$ and $\mathbf{p}_1(fe) \downarrow \Vdash \psi(x, y, u)$.

We construct $b^* = \{\langle e, y \rangle : e \in \mathcal{A} \wedge \exists d \in \mathcal{A} \langle d, y \rangle \in b\}$. Now, for any $\langle e, x \rangle \in a$, there exists $\langle e, y \rangle \in b^*$ such that $\mathbf{p}_1(fe) \downarrow \Vdash \psi(x, y, u)$. By subset collection in the base theory, there is some set c , depending only on a and b^* , such that for any $u \in V(\mathcal{A})$ and realiser f as above, there is some $d \in c$ such that

- (i) for any $\langle e, x \rangle \in a$, there is $\langle e, y \rangle \in d$ such that $\mathbf{p}_1(fe) \Vdash \psi(x, y, u)$, and
- (ii) for any $v \in d$, we have $v = \langle e, y \rangle \in b^*$ and there is $\langle e, x \rangle \in a$ such that $\mathbf{p}_1(fe) \Vdash \psi(x, y, u)$.

Condition (ii) ensures that such $d \subseteq b^*$ and thus we can safely restrict

$$c^* = \{\langle 0, d \rangle : d \in c, d \subseteq b^*\} \in V(\mathcal{A})$$

by Δ_0 -separation and then collection. Just as in the proof of Lemma 4.2.9, given any $u \in V(\mathcal{A})$, $\lambda e. \mathbf{pe}(\mathbf{p}_1(fe))$ realises both $\forall x \in a \exists y \in d \psi(x, y, u)$ and $\forall y \in d \exists x \in a \psi(x, y, u)$ for the specific d above, which satisfies $\langle 0, d \rangle \in c^*$. It

follows that

$$\lambda f. \mathbf{p}0(\mathbf{p}(\lambda e. \mathbf{p}e(\mathbf{p}_1(fe)))(\lambda e. \mathbf{p}e(\mathbf{p}_1(fe))))$$

realises the axiom. $\square$

Finally, for Δ_0 -separation, we shall avoid the complexities of analysing the realisability condition for a Δ_0 -formula by invoking the following result:

Let the *binary intersection* axiom be the statement that

$$\forall a \forall b \exists y \forall x (x \in y \leftrightarrow x \in a \wedge x \in b),$$

which essentially states that the intersection $a \cap b$ is a set. Then we have:

Theorem 4.2.11 ([3, Corollary 9.5.7]). *The system IKP – Δ_0 -separation + strong collection (6') suffices³⁵ to prove that Δ_0 -separation is equivalent to the binary intersection axiom.*

Proof sketch. The proof is quite technical and we refer the readers to [3, section 9.5] for details. Roughly speaking, we observe that given a set $u = \{x : x = a \wedge \varphi\}$ where x does not occur free in φ , strong collection implies that for any set b ,

$$\{x : x = b \wedge \varphi\} = \{b : x \in u\}$$

exists. Thus, if we let $!\varphi$ denote the statement $\forall a \exists b b = \{x : x = a \wedge \varphi\}$, then we just need to find one such set in order to prove $!\varphi$.

Now, notice that for any a, b ,

$$\{a\} \cap \{b\} = \{x : x = a \wedge a = b\}.$$

Thus the binary intersection axiom implies that $!(a = b)$. This is used as a starting point to derive $!\varphi$ for every Δ_0 -formula φ . It then follows that

$$\{x \in a : \varphi(x)\} = \bigcup \{\{y : y = x \wedge \varphi(x)\} : x \in a\}$$

by strong collection and union axioms. $\square$

It follows that we just need to realise the binary intersection axiom:

³⁵To be exact, strong Δ_0 -collection already suffices for the constructions here.

Lemma 4.2.12. *The binary intersection axiom is realised:*

$$\forall a \forall b \exists y \forall x (x \in y \leftrightarrow x \in a \wedge x \in b).$$

Proof. Given $a, b \in V(\mathcal{A})$, consider

$$c = \{\langle \mathbf{p}fe, x \rangle : \langle f, x \rangle \in a, e \Vdash x \in b\} \in V(\mathcal{A}).$$

Note that $c = \bigcup \{\{\langle \mathbf{p}fe, x \rangle : e \in F_\in(x, b)\} : \langle f, x \rangle \in a\}$ and hence a set. Observe that

$$(\mathbf{p}(\mathbf{p}fg)d \Vdash x \in c) \leftrightarrow (\mathbf{p}fd \Vdash x \in a) \wedge (\mathbf{p}(\mathbf{p}_0g)(\iota_t d(\mathbf{p}_1g)) \Vdash x \in b),$$

and thus

$$\begin{aligned} \lambda d. \mathbf{p}(\mathbf{p}(\mathbf{p}_{00}d)(\mathbf{p}_1d))(\mathbf{p}(\mathbf{p}_{010}d)(\iota_t(\mathbf{p}_1d)(\mathbf{p}_{011}d))) \Vdash x \in c \rightarrow x \in a \wedge x \in b, \\ \lambda d. \mathbf{p}(\mathbf{p}(\mathbf{p}_{00}d)(\iota_{s1}(\mathbf{p}_{01}d)(\mathbf{p}_1d))) (\mathbf{p}_{01}d) \Vdash x \in a \wedge x \in b \rightarrow x \in c \end{aligned}$$

as desired. □

Combining the results so far and Proposition 2.3.1, we can conclude that:

Theorem 4.2.13 ([31, Theorem 5.1]). *Let φ be any theorem of CZF, then CZF proves $V(\mathcal{A}) \models \varphi$, i.e. there exists $d \in \mathcal{A}$ such that $d \Vdash \varphi$.*

4.3 Intuitionistic L

Just as in classical set theory, we say that a set x is *transitive* if every element of x is also a subset of x , i.e.

$$\forall y \in x \forall z \in y \ z \in x,$$

and we define the *ordinals*, Ord , as the class of transitive sets with transitive elements.

However, many important properties of ordinals in classical set theory rely on the law of excluded middle, and hence cannot be recovered in intuitionistic contexts. For example, we have:

Lemma 4.3.1. *The realisability model of CZF in CZF realises that*

$$\neg \forall \alpha, \beta \in \text{Ord} (\alpha \subseteq \beta \rightarrow \alpha \in \beta \vee \alpha = \beta).$$

Proof. It suffices to find a counterexample, i.e. $\alpha, \beta \in V(\mathcal{A})$ such that $\alpha \in \text{Ord}$, $\beta \in \text{Ord}$ and $\alpha \subseteq \beta$ are realised, but no $d \in \mathcal{A}$ realises $\alpha \in \beta$ or $\alpha = \beta$. We shall take

$$\alpha = \bar{2} = \{\langle 0, \bar{0} \rangle, \langle 1, \bar{1} \rangle\}, \quad \beta = \{\langle 0, \bar{0} \rangle, \langle 0, \bar{1} \rangle\}.$$

It is easy to verify that $\forall n \in \omega \, n \in \text{Ord}$ still holds in CZF (or even IKP), hence $\alpha \in \text{Ord}$ must be realised. In the case of β , simply observe that if $\langle a, x \rangle \in \beta$ and $\langle b, y \rangle \in x$, then we must have $x = \bar{1}$ and $y = \emptyset$, so $\mathbf{p}0(\mathbf{p}00) \Vdash y \in \beta$ and thus $\lambda ab. \mathbf{p}0(\mathbf{p}00) \Vdash \beta$ is transitive. Since both $\bar{0}$ and $\bar{1}$ contain no elements other than empty sets (as seen in the realisability model), $\lambda a. 0$ trivially realises that both are transitive. Putting them together,

$$\mathbf{p}(\lambda ab. \mathbf{p}0(\mathbf{p}00))(\lambda xa. 0) \Vdash \beta \in \text{Ord}.$$

Now, it is easy to see that $\lambda d. \mathbf{p}0\iota_r \Vdash \alpha \subseteq \beta$. We also observe that no $d \in \mathcal{A}$ can realise $a = \bar{0}$ where $a \in V(\mathcal{A})$ is inhabited, thus no d can realise $\alpha = \bar{0}$, nor $\alpha = \bar{1}$ (in the latter case, since $\langle 1, \bar{1} \rangle \in \alpha$ and the sole element of $\bar{1}$ is $\langle 0, \bar{0} \rangle$, thus $\mathbf{p}_1(\mathbf{p}_0 d 1)$ will then realise $\bar{1} = \bar{0}$). Therefore, no d can realise $\alpha \in \beta$.

Finally, suppose that some $f \Vdash \beta \subseteq \alpha$. Using the fact that no d realises $\bar{0} = \bar{1}$, we know that f must match up $\bar{0}$ with $\bar{0}$ and $\bar{1}$ with $\bar{1}$, i.e. $\mathbf{p}_0(f0) = 0$ for $\langle 0, \bar{0} \rangle \in \beta$ and $\mathbf{p}_0(f0) = 1$ for $\langle 0, \bar{1} \rangle \in \beta$. But then $0 = 1$ in the base theory, a contradiction. We conclude that $\alpha = \beta$ is not realised either. $\square$

Intuitionistic set theory also does not prove the trichotomy that every ordinal is either zero, a successor or a limit — simply see β in the proof above and observe that it is not a successor.

Thus, the traditional definition for transfinite hierarchies like Gödel's constructible universe L does not work, and we need to define them using a uniform clause instead. Namely, it is conventional to stipulate

$$L_\alpha = \bigcup_{\beta \in \alpha} \text{def}(L_\beta)$$

for all $\alpha \in \text{Ord}$ in intuitionistic settings, and then take the union $L = \bigcup_{\alpha \in \text{Ord}} L_\alpha$ as usual. Here $\text{def}(x)$ still intends to mean the definable subsets of the first-order structure $\langle x; \in \rangle$ in the language of set theory.

Robert Lubarsky [24] first studied intuitionistic L and checked that construc-

tions related to $\text{def}(x)$ work as expected in intuitionistic set theory. Essentially, we consider the set of finite tuples

$$x^{<\omega} = \bigcup_{n \in \omega} x^n$$

and construct the truth predicates $\text{Tr}_{\ulcorner \varphi \urcorner}$, where $\ulcorner \varphi \urcorner \in \omega$ is the Gödel numbering of formula φ , internally as³⁶

$$\begin{aligned} \text{Tr}_{\ulcorner x_i \in x_j \urcorner} &= \{ \tau \in x^{<\omega} : |\tau| > i, j, \tau_i \in \tau_j \}, \\ \text{Tr}_{\ulcorner x_i = x_j \urcorner} &= \{ \tau \in x^{<\omega} : |\tau| > i, j, \tau_i = \tau_j \}, \\ \text{Tr}_{\ulcorner \varphi \wedge \psi \urcorner} &= \text{Tr}_{\ulcorner \varphi \urcorner} \cap \text{Tr}_{\ulcorner \psi \urcorner}, \\ \text{Tr}_{\ulcorner \varphi \vee \psi \urcorner} &= \text{Tr}_{\ulcorner \varphi \urcorner} \cup \text{Tr}_{\ulcorner \psi \urcorner}, \\ \text{Tr}_{\ulcorner \neg \varphi \urcorner} &= x^{<\omega} \setminus \text{Tr}_{\ulcorner \varphi \urcorner}, \\ \text{Tr}_{\ulcorner \varphi \rightarrow \psi \urcorner} &= \{ \tau \in x^{<\omega} : \tau \in \text{Tr}_{\ulcorner \varphi \urcorner} \rightarrow \tau \in \text{Tr}_{\ulcorner \psi \urcorner} \}, \\ \text{Tr}_{\ulcorner \forall x_i \varphi \urcorner} &= \{ \tau \in x^{<\omega} : |\tau| > i, \forall y \in x \tau[y/i] \in \text{Tr}_{\ulcorner \varphi \urcorner} \}, \\ \text{Tr}_{\ulcorner \exists x_i \varphi \urcorner} &= \{ \tau \in x^{<\omega} : |\tau| > i, \exists y \in x \tau[y/i] \in \text{Tr}_{\ulcorner \varphi \urcorner} \} \end{aligned}$$

by Σ -recursion on natural numbers, where $\tau[y/i]$ denotes the result of replacing the i^{th} component of finite tuple τ by object y . Each $\text{Tr}_{\ulcorner \varphi \urcorner} \subseteq x^{<\omega}$ is a set by Δ_0 -separation. One can then define

$$\text{def}(x) = \{ \{ y \in x : y \frown \tau \in \text{Tr}_{\ulcorner \varphi \urcorner} \} : \varphi \text{ is a formula, } \tau \in x^{<\omega} \}$$

by Δ_0 -separation and collection.

IKP suffices for this construction, where we observe that the axiom of infinity is crucial for encoding the set of first-order formulae and thus the recursive process. Recently in [25], Matthews and Rathjen studied the alternative formulation of L using Barwise's fundamental operations in [4, section II.6] in absence of the axiom of infinity, and verified that the two approaches still coincide in intuitionistic set theory. For the topics examined in this thesis, however, Lubarsky's construction suffices and is more convenient.

A first observation about intuitionistic L is that the following classical proof makes an essential use of the property invalidated in Lemma 4.3.1:

³⁶We take a simplified approach with the guarding conditions on the length of τ here, which says precisely that when evaluating a formula φ on a finite set of variable assignments τ , any atomic or quantifier subformula containing a variable not occurring in the assignment is simply treated as $\perp$ — even for bound occurrences. It is easy to see that this semantic simplification does not affect the expressive power of first-order logic.

Lemma 4.3.2 (ZF or KP). *For any $\alpha \in \text{Ord}$, $L_\alpha \cap \text{Ord} = \alpha$.*

Proof. We prove this by set induction on α . By the inductive hypothesis, $L_\beta \cap \text{Ord} = \beta$ for any $\beta \in \alpha$. Since Ord is a Δ_0 -class, it follows that $\beta \in \text{def}(L_\beta) \subseteq L_\alpha$ and hence $\alpha \subseteq L_\alpha \cap \text{Ord}$.

For the converse, suppose that $\gamma \in L_\alpha \cap \text{Ord}$. There is some $\beta \in \alpha$ such that $\gamma \in \text{def}(L_\beta)$, which means $\gamma \subseteq L_\beta \cap \text{Ord} = \beta$. Using the classical fact that

$$\forall \alpha, \beta \in \text{Ord} (\alpha \subseteq \beta \rightarrow \alpha \in \beta \vee \alpha = \beta),$$

it follows that $\gamma \in \beta \vee \gamma = \beta$, and thus $\gamma \in \alpha$. $\square$

In section 4.4, we will explain a formal proof of the fact that the opposite is indeed consistent with CZF. This implies that there is no longer a canonical way to make every $\alpha \in \text{Ord}$ a definable set — thus an element — in L . The following question, first raised by Lubarsky in [24], remains open:

Problem 4.3.3. Does any non-classical system of set theory prove $\text{Ord} \subseteq L$?

On the other hand, Lubarsky showed in [24] that several other properties of classical L can still be recovered in intuitionistic logic, even though the proof can be much more complicated than in the classical case. One key result is that:

Theorem 4.3.4. *Any Δ_0 -formula φ is decidable on L_ω , i.e. let $x_1, \dots, x_k$ enumerate the free variables in φ , then CZF (or even IKP) proves*

$$\forall x_1, \dots, x_k \in L_\omega (\varphi \vee \neg \varphi).$$

Proof sketch. By simultaneous induction on ω , one can still show the trichotomy for natural numbers

$$\forall n, m \in \omega (n \in m \vee n = m \vee m \in n),$$

and this decidability is preserved by logical connectives and natural-number-bounded quantifiers. It follows that any Δ_0 -formula evaluated in the structure $\langle \omega; \in \rangle$, which can possibly contain Σ_1 -definable functions such as arithmetic operations, satisfies excluded middle.

Now, we build the (reverse) Ackermann encoding, a function f on domain ω , given as

$$f(n) = \{f(m) : m \in n, n \bmod 2^{m+1} \geq 2^m\}.$$

One can verify by induction on $n \in \omega$ that every L_n is precisely the image of f on an appropriate initial segment of ω , because given the inductive hypothesis, the map f then allows us to translate any formula evaluated on L_n into a Δ_0 -formula evaluated on ω , which is decidable.

Through this induction, we see that f is a bijection between ω and L_ω . Thus, every Δ_0 -formula on L_ω translates into a Δ_0 -formula with arithmetic operations on ω , and is decidable as desired.

We refer the readers to [24] for more detail. $\square$

Corollary 4.3.5. *For any $n \in \omega$, $L_n \cap \text{Ord} = n$. It follows immediately that $L_\omega \cap \text{Ord} = \omega$.*

Proof. Suppose that $\alpha \in L_\omega$ and $n \in \omega \cap L_\omega$ satisfies $\alpha \subseteq n$. We know from Theorem 4.3.4 that

$$\exists k \in n \, k \notin \alpha \vee \forall k \in n \, k \in \alpha.$$

The latter disjunct implies $\alpha = n$; in the former case, we show that $\alpha \subseteq k$. Suppose that this is not the case, i.e. there is some $\ell \in \alpha$ such that $\ell \not\subseteq k$. However, both $k, \ell \in \omega$, thus by the trichotomy for natural numbers, we have either $k \in \ell$ or $k = \ell$, both contradicting $k \notin \alpha$. Thus we must have $\alpha \subseteq k$ by decidability from Theorem 4.3.4.

Hence, by induction on $n \in \omega$, we see that

$$\forall n \in \omega \cap L_\omega \, \forall \alpha \in L_\omega \, (\alpha \subseteq n \rightarrow \alpha \in n \vee \alpha = n).$$

The proof of Lemma 4.3.2 now applies to give $\forall n \in \omega \, L_n \cap \text{Ord} = n$. $\square$

Another result of Lubarsky [24] that is relevant is the following:

Theorem 4.3.6. *The relativised statement $(V = L)^L$ holds.*

Note that this is proved without assuming that $\text{Ord} \subseteq L$, unlike in classical set theory. For any $\alpha \in \text{Ord}$, Lubarsky constructed an ordinal

$$\alpha^* = \left\{ \beta \in \text{Ord} : L_{(\alpha_{\text{aug}} +_{\text{H}} n)^-} \models \exists x (x = L_\beta \wedge \text{def}(x) \subseteq L_\alpha) \right\},$$

where $\alpha_{\text{aug}} = \bigcup \{ \beta_{\text{aug}} : \beta \in \alpha \} \cup \{ \alpha \} \cup \{ \omega^+ \}$, $(\alpha_{\text{aug}} +_{\text{H}} n)^-$ is a variation of *hereditary addition* (cf. section 5.1 for normal ordinal addition) given in terms

of the standard variant as

$$\begin{aligned}\alpha +_{\mathbf{H}} \gamma &= \left(\alpha \cup \bigcup_{\beta \in \alpha} (\beta +_{\mathbf{H}} \gamma) \right) + \gamma, \\ (\alpha +_{\mathbf{H}} \gamma)^- &= \alpha \cup \bigcup_{\beta \in \alpha} (\beta +_{\mathbf{H}} \gamma),\end{aligned}$$

and $n \in \omega$ is fixed large enough for technical existential witnesses. Since the only parameter L_α itself is a set in L , it follows that $\alpha^* \in L \cap \text{Ord}$ and Lubarsky verified that $L_{\alpha^*} = L_\alpha$ to derive $L \models V = L$.

4.4 Some exotic ordinals and a modality

Before ending this chapter, we will reason with L inside the realisability model in section 4.2 to formally demonstrate that intuitionistic L can indeed have wild behaviours beyond the decidable segment L_ω . We shall show that CZF is consistent with the statement $\exists \alpha \in \text{Ord} \neg L_\alpha \cap \text{Ord} = \alpha$. We also see that the idea above of ‘constructing α^* so that $L_{\alpha^*} = L_\alpha$ ’ is non-trivial, as it is consistent to have $\alpha \neq \beta$ yet $L_\alpha = L_\beta$.

To begin, we construct a family of names for ‘ ω -like’ intuitionistic ordinals. For any $u \subseteq \omega$ in the base theory, we denote

$$\bar{\omega}_{(u)} = \{ \langle n, \bar{n} \rangle : n \in \omega \} \cup \{ \langle 0, \bar{n} \rangle : n \in u \} \in V(\mathcal{A}).$$

Notice that whether $0 \in u$ is irrelevant, as $\bar{\omega}_{(u)} = \bar{\omega}_{(u \cup \{0\})}$. Thus in most scenarios we will just work with sets $u \subseteq \omega$ satisfying $0 \in u$.

Lemma 4.4.1. *For any $u \subseteq \omega$, $V(\mathcal{A}) \models \bar{\omega}_{(u)} \in \text{Ord}$.*

Proof. Simply observe that $\lambda ab. \mathbf{pb}_{\iota_r} \Vdash \bar{\omega}_{(u)}$ is transitive, where the same realiser also realises $\bar{n}$ is transitive uniformly for any $n \in \omega$. Thus

$$\mathbf{p}(\lambda ab. \mathbf{pb})(\lambda xab. \mathbf{pb}_{\iota_r}) \Vdash \bar{\omega}_{(u)} \in \text{Ord}. \quad \square$$

Lemma 4.4.2. *For any $u, v \subseteq \omega$ such that $0 \in u, v$, we have $V(\mathcal{A}) \models \bar{\omega}_{(u)} \subseteq \bar{\omega}_{(v)}$ if and only if $u \subseteq v$.*

Proof. The backward direction is immediate, as $\lambda n. \mathbf{pn}_{\iota_r}$ realises $\bar{\omega}_{(u)} \subseteq \bar{\omega}_{(v)}$ when $u \subseteq v$ holds.

For the forward direction, observe by induction on ω that for any $n, m \in \omega$, $V(\mathcal{A}) \models \bar{n} = \bar{m}$ if and only if $n = m$. Now, suppose that some $f \in \mathcal{A}$ realises $\bar{\omega}_{(u)} \subseteq \bar{\omega}_{(v)}$. The sole index $i \in \omega$ such that $\langle i, \bar{0} \rangle \in \bar{\omega}_{(u)}$ is $i = 0$, and the same holds for $\bar{\omega}_{(v)}$. Thus, f must match these indices up, i.e. $\mathbf{p}_0(f0) = 0$. It follows that for any $n \in u$, i.e. $\langle 0, \bar{n} \rangle \in \bar{\omega}_{(u)}$, there is some $\langle j, \bar{m} \rangle \in \bar{\omega}_{(v)}$ such that

$$j = \mathbf{p}_0(f0) = 0,$$

and $\mathbf{p}_1(f0) \Vdash \bar{n} = \bar{m}$, i.e. $m = n$. Hence $\langle 0, \bar{n} \rangle \in \bar{\omega}_{(v)}$.

It follows that for any $n \in u$, if $n \neq 0$ then $n \in v$. Using the assumption that $0 \in u, v$, we have $u \subseteq v$. $\square$

Most notably, $\bar{\omega}_{(\emptyset)} = \bar{\omega}_{(\{0\})} = \bar{\omega}$ is the least element in this family. In this section, we will focus on the behaviour of the opposite extreme, $\bar{\omega}_{(\omega)}$.

Realisability with a modal operator

To completely characterise $\bar{\omega}_{(\omega)}$, we need to expand the formal language and introduce a modal operator $\Box$, so that $V(\mathcal{A}) \models \phi \rightarrow \Box\psi$ means roughly ‘given a realiser $d \Vdash \phi$, I know that there will be a realiser for ψ , but I do not know how to compute it from d ’. Strictly speaking, this is neither an instance of necessity nor possibility (cf. Lemma 4.4.3 below), but more closely related to Kosta Došen’s system of modal logic for the intuitionistic double negation in [11], from which I took the inspiration.

Formally, we simply expand the first-order language by a new clause that, whenever φ is a formula in the modal language, so is $\Box\varphi$. We extend the realisability conditions in section 4.2 by

$$d \Vdash \Box\varphi \quad :\Longleftrightarrow \quad \exists e \in \mathcal{A} \, e \Vdash \varphi.$$

Lemma 4.4.3. *The following modal postulates are realised for any formulae φ and ψ :*

- (1) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$;
- (2) $\varphi \rightarrow \Box\varphi$;
- (3) $\Box\varphi \rightarrow \neg\neg\varphi$.

Proof. Notice nothing needs to be computed to realise any assertion of the

form $\Box\varphi$ that is ‘correct’ (just like negated assertions). Thus:

- (1) $\lambda de.0$ realises $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$;
- (2) $\lambda d.0$ realises $\varphi \rightarrow \Box\varphi$;
- (3) $\lambda d.0$ realises $\Box\varphi \rightarrow \neg\neg\varphi$. □

If the underlying base theory is classical, e.g. when the aforementioned realisability of CZF is performed in the base theory ZF, we additionally realise $\Box\varphi \leftrightarrow \neg\neg\varphi$ trivially, since classically,

$$(\exists e \in \mathcal{A} e \Vdash \varphi) \leftrightarrow (\neg \forall e \in \mathcal{A} \neg e \Vdash \varphi).$$

Then we recover precisely the modality considered in Došen [11].

Meanwhile, returning to the base theory CZF, we check that:

Proposition 4.4.4. *It is realised that*

$$x \in \bar{\omega}_{(\omega)} \leftrightarrow (x \subseteq \bar{\omega} \wedge \Box x \in \bar{\omega}) \wedge \forall y \in \bar{\omega} (\Box y \in x \rightarrow y \in x).$$

Proof. For the forward direction, we just need to equivalently realise

$$\forall x \in \bar{\omega}_{(\omega)} (x \subseteq \bar{\omega} \wedge \Box x \in \bar{\omega} \wedge \forall y \in \bar{\omega} (\Box y \in x \rightarrow y \in x)).$$

Observe that $\lambda n. \mathbf{p}n\iota_r \Vdash \bar{m} \subseteq \bar{\omega}$ for any $m \in \omega$, thus $\lambda xn. \mathbf{p}n\iota_r \Vdash \forall x \in \bar{\omega}_{(\omega)} x \subseteq \bar{\omega}$. Next, $\lambda x.0$ trivially realises $\forall x \in \bar{\omega}_{(\omega)} \Box x \in \bar{\omega}$, since every element in $\bar{\omega}_{(\omega)}$ is of the form $\langle n, \bar{m} \rangle$. Finally, for any $n, m \in \omega$, $\lambda d. \mathbf{p}n\iota_r \Vdash \Box \bar{n} \in \bar{m} \rightarrow \bar{n} \in \bar{m}$, thus

$$\lambda xnd. \mathbf{p}n\iota_r \Vdash \forall x \in \bar{\omega}_{(\omega)} \forall y \in \bar{\omega} (\Box y \in x \rightarrow y \in x).$$

Now, for the backward direction, fix $x \in V(\mathcal{A})$ and suppose that f realises the entire right-hand side of the biconditional. It follows from the realisability of the modality that there exists some $d \in \mathcal{A}$ such that $d \Vdash x \in \bar{\omega}$, i.e. $\mathbf{p}_1 d \Vdash x = \overline{\mathbf{p}_0 d}$. We derive from $\mathbf{p}_{00} f \Vdash x \subseteq \bar{\omega}$ that for any $\langle g, y \rangle \in x$, $\mathbf{p}_{00} f g \Vdash y \in \bar{\omega}$, i.e.

$$\mathbf{p}_1(\mathbf{p}_{00} f g) \Vdash y = \bar{m}$$

where $m = \mathbf{p}_0(\mathbf{p}_{00} f g)$. Using that $x = \overline{\mathbf{p}_0 d}$ is realised with the fact that $\bar{m} \in \bar{n}$ is realised if and only if $m \in n$, we must have $m \in \mathbf{p}_0 d$ and thus $\mathbf{p}_{00} f g$ also realises $y \in \overline{\mathbf{p}_0 d}$, i.e. $\mathbf{p}_{00} f \Vdash x \subseteq \overline{\mathbf{p}_0 d}$.

On the other hand, since $\mathbf{p}_1 d \Vdash x = \overline{\mathbf{p}_0 d}$, for any $m \in \mathbf{p}_0 d$, $\mathbf{p}_{11} dm \Vdash \bar{m} \in x$. It

follows that $0 \Vdash \Box \bar{m} \in x$ and thus $\mathbf{p}_1 f m 0 \Vdash \bar{m} \in x$. Therefore, $\lambda m. \mathbf{p}_1 f m 0 \Vdash \overline{\mathbf{p}_0 d} \subseteq x$, and we can conclude that

$$\mathbf{p}(\mathbf{p}_{00}f)(\lambda m. \mathbf{p}_1 f m 0) \Vdash x = \overline{\mathbf{p}_0 d}$$

where the realiser does not involve d . Since $\langle 0, \overline{\mathbf{p}_0 d} \rangle \in \bar{\omega}_{(\omega)}$, we have

$$\begin{aligned} & \lambda f. \mathbf{p}0(\mathbf{p}(\mathbf{p}_{00}f)(\lambda m. \mathbf{p}_1 f m 0)) \\ & \Vdash (x \subseteq \bar{\omega} \wedge \Box x \in \bar{\omega}) \wedge \forall y \in \bar{\omega} (\Box y \in x \rightarrow y \in x) \rightarrow x \in \bar{\omega}_{(\omega)}. \quad \square \end{aligned}$$

This will be useful as the defining formula for $\bar{\omega}_{(\omega)}$ when we reason inside $V(\mathcal{A})$. However, we cannot use a formula containing modal operators to show definability inside L , thus we see that one of the $\Box$'s can be dropped:

Lemma 4.4.5. $V(\mathcal{A}) \models \forall \alpha \in \bar{\omega}_{(\omega)} \forall n \in \bar{\omega} (\neg \neg n \in \alpha \rightarrow \Box n \in \alpha)$.

Proof. We use the fact that CZF proves the trichotomy for natural numbers, which implies

$$\forall m \in \omega \forall n \in \omega (\neg \neg n \in m \rightarrow n \in m).$$

Using the modal postulates in Lemma 4.4.3 (and Lemma 4.2.7 that $\bar{\omega}$ is indeed the least inductive set), it follows that $V(\mathcal{A})$ realises

$$\Box m \in \bar{\omega} \rightarrow (\Box n \in \bar{\omega} \rightarrow (\Box \neg \neg n \in m \rightarrow \Box n \in m)).$$

Working inside $V(\mathcal{A})$, we consider any $\alpha \in \bar{\omega}_{(\omega)}$. It follows from Proposition 4.4.4 that $\Box \alpha \in \bar{\omega}$. For any $n \in \bar{\omega}$, suppose $\neg \neg n \in \alpha$, then we have $\Box n \in \bar{\omega}$ and $\Box \neg \neg n \in \alpha$ by Lemma 4.4.3, and thus $\Box n \in \alpha$ as desired. $\square$

Corollary 4.4.6. *It is realised that*

$$x \in \bar{\omega}_{(\omega)} \leftrightarrow (x \subseteq \bar{\omega} \wedge \Box x \in \bar{\omega}) \wedge \forall y \in \bar{\omega} (\neg \neg y \in x \rightarrow y \in x).$$

Proof. Using Proposition 4.4.4 as a basis, for the forward direction, it suffices to derive $\Box y \in x$ given $x \in \bar{\omega}_{(\omega)}$ and $y \in \bar{\omega}$ such that $\neg \neg y \in x$ holds. This follows from Lemma 4.4.5.

For the backward direction, given $x \subseteq \bar{\omega}$ such that $\Box x \in \bar{\omega}$, it suffices to derive $\forall y \in \bar{\omega} (\Box y \in x \rightarrow y \in x)$ from $\forall y \in \bar{\omega} (\neg \neg y \in x \rightarrow y \in x)$. This follows from the modal postulate $\Box \varphi \rightarrow \neg \neg \varphi$ in Lemma 4.4.3. $\square$

Before relating $\bar{\omega}_{(\omega)}$ to the constructible universe L , we need a technical lemma:

Lemma 4.4.7 (IKP). *L -ranks can be defined for elements in L_ω , i.e. for any $x \in L_\omega$, there exists a natural number $k \in \omega$ such that*

$$\forall \ell \in \omega (x \in L_\ell \leftrightarrow k \in \ell).$$

Proof. For any $x \in L_\omega$, there is $n \in \omega$ such that $x \in \text{def}(L_n)$. We show

$$\forall x \in \text{def}(L_n) \exists k \in \omega \forall \ell \in \omega (x \in L_\ell \leftrightarrow k \in \ell)$$

by induction on n .

When $x \in \text{def}(L_0)$, then $x \subseteq L_0 = \emptyset$ and thus $x = \emptyset$. Clearly we have $\forall \ell \in \omega (x \in L_\ell \leftrightarrow 0 \in \ell)$. Otherwise, $n = m^+$ is a successor, and for $x \in \text{def}(L_n)$, by Theorem 4.3.4 we know that either $x \in L_n = \text{def}(L_m)$ or $x \in \text{def}(L_n) \setminus L_n$. The former case is covered by the inductive hypothesis, whereas in the latter case, we use the trichotomy for natural numbers to see that

$$\forall \ell \in \omega (x \in L_\ell \leftrightarrow n \in \ell),$$

since $n \notin \ell$ implies $\ell \subseteq n$ and thus $x \notin L_\ell \subseteq L_n$. $\square$

Corollary 4.4.8. $V(\mathcal{A}) \models \forall \alpha \in \bar{\omega}_{(\omega)} \forall x \in L_{\bar{\omega}} (\Box x \in L_\alpha \rightarrow x \in L_\alpha)$.

Proof. Work inside $V(\mathcal{A})$ and consider any $\alpha \in \bar{\omega}_{(\omega)}$ and any $x \in L_{\bar{\omega}}$. By Lemma 4.4.7, we can find $k \in \bar{\omega}$ such that $\forall \ell \in \bar{\omega} (x \in L_\ell \leftrightarrow k \in \ell)$, thus from the modal postulates,

$$\Box \ell \in \bar{\omega} \rightarrow (\Box x \in L_\ell \leftrightarrow \Box k \in \ell).$$

We have $\Box \alpha \in \bar{\omega}$ by Proposition 4.4.4, so assuming $\Box x \in L_\alpha$, we can derive $\Box k \in \alpha$. The last conjunct in Proposition 4.4.4 then implies that $k \in \alpha$. It follows that $k^+ \subseteq \alpha$ and we have $x \in L_{k^+} \subseteq L_\alpha$. $\square$

A family of coinciding $L_{\bar{\omega}_{(u)}}$'s

Consider a Δ_0 -formula $\varphi(x, y)$ in the language of set theory — without the modal operator $\Box$ — with no other free variables, such that

$$\text{CZF} \vdash \forall x \in \omega \exists! y \in L_\omega \varphi(x, y).$$

We say that a set $u \subseteq \omega$ *encompasses* ω *through* φ if $0 \in u$ and, for any $n \in \omega$, there exists $m \in u$ such that

$$V(\mathcal{A}) \models \{x \in L_{\bar{m}} \cap \text{Ord} : \exists y \in L_{\bar{m}} \varphi(x, y)\} = \bar{n}.$$

Informally speaking, this definition means that all elements of ω can be defined from elements of u in a uniform way that is easy to implement in L . Clearly ω encompasses itself by taking

$$\varphi(x, y) \quad :\Longleftrightarrow \quad x = x,$$

so that $V(\mathcal{A}) \models \{x \in L_{\bar{n}} \cap \text{Ord} : \exists y \in L_{\bar{n}} \varphi(x, y)\} = \bar{n}$ for all $n \in \omega$. We first see that there are many other subsets of ω that encompass ω :

Lemma 4.4.9. *Let $u \subseteq \omega$ be any subset containing all the even numbers. Any such u encompasses ω through the same formula.*

Proof. Take $\varphi(x, y)$ to express the following: x is an ordinal and y is a function on domain x^+ , satisfying $y(\emptyset) = x$ and $y(z^+) = y(z)^+$ for every $z \in x$. By induction on ω , such y exists and must be unique for any $x \in \omega$. Given the definition of Kuratowski pairs, we have

$$x \cdot 2 \in \{x, x \cdot 2\} \in \langle x, x \cdot 2 \rangle \in y.$$

It follows that

$$\forall n \in \omega \{x \in L_{n \cdot 2 + 2} \cap \text{Ord} : \exists y \in L_{n \cdot 2 + 2} \varphi(x, y)\} = n$$

is a theorem in CZF. Since it is easy to check that $V(\mathcal{A})$ realises the correct arithmetic relations between names $\bar{n}$ and $\overline{n \cdot 2 + 2}$, we must have

$$V(\mathcal{A}) \models \{x \in L_{\overline{n \cdot 2 + 2}} \cap \text{Ord} : \exists y \in L_{\overline{n \cdot 2 + 2}} \varphi(x, y)\} = \bar{n}. \quad \square$$

We now define a new formula

$$\Psi(x) \quad :\Longleftrightarrow \quad x \in \text{Ord} \wedge \forall y \in \omega (\neg \neg y \in x \rightarrow y \in x)$$

where, pedantically, we do not want to think of ω as a parameter to this formula, but rather read the quantifier $\forall y \in \omega$ as letting y range over the Δ_0 -class for the least inductive set given in Lemma 4.1.1. We verify that:

Theorem 4.4.10. *For any $u \subseteq \omega$ that encompasses ω through some Δ_0 -formula, we always realise*

$$\bar{\omega}_{(\omega)} = \left\{ x \in L_{\bar{\omega}_{(u)}} : L_{\bar{\omega}_{(u)}} \models \Psi(x) \right\} \in \text{def}(L_{\bar{\omega}_{(u)}}).$$

Proof. First note that $\bar{\omega} \subseteq \bar{\omega}_{(u)}$ by Lemma 4.4.2, thus $\bar{\omega} \subseteq L_{\bar{\omega}_{(u)}}$ by Corollary 4.3.5, where $L_{\bar{\omega}_{(u)}}$ is a transitive set. Thus indeed we can define ω as a Δ_0 -definable subclass of $L_{\bar{\omega}_{(u)}}$ by Lemma 4.1.1 and avoid using it as a parameter in $\Psi(x)$.

Now, for the forward direction, first observe that our assumption about u encompassing ω implies that, for some Δ_0 -formula $\varphi(x, y)$,

$$\lambda a. \mathbf{p00} \Vdash \forall \alpha \in \bar{\omega}_{(\omega)} \exists \beta \in \bar{\omega}_{(u)} \Box \{x \in L_\beta \cap \text{Ord} : \exists y \in L_\beta \varphi(x, y)\} = \alpha.$$

Work inside $V(\mathcal{A})$ and consider any $\alpha \in \bar{\omega}_{(\omega)}$. By Lemma 4.4.1 and Corollary 4.4.6, we know that $\Psi(\alpha)$ holds, thus it suffices to show that $\alpha \in L_{\bar{\omega}_{(u)}}$. Our assumption for u now means there is some $\beta \in \bar{\omega}_{(u)}$ such that $\Box \{x \in L_\beta \cap \text{Ord} : \exists y \in L_\beta \varphi(x, y)\} = \alpha$ holds and we know

$$\{x \in L_\beta \cap \text{Ord} : \exists y \in L_\beta \varphi(x, y)\} \in \text{def}(L_\beta) \subseteq L_{\bar{\omega}_{(u)}}.$$

Thus it suffices to show that actually $\{x \in L_\beta \cap \text{Ord} : \exists y \in L_\beta \varphi(x, y)\} = \alpha$.

We take any $x \in L_\beta \cap \text{Ord}$ such that there exists $y \in L_\beta$ satisfying $\varphi(x, y)$, then the modal postulates imply $\Box x \in \alpha$. Observe that $\beta \in \bar{\omega}_{(u)} \subseteq \bar{\omega}_{(\omega)}$ by Lemma 4.4.2 and thus $\beta \subseteq \omega$ by Proposition 4.4.4. Hence,

$$x \in L_\beta \cap \text{Ord} \subseteq L_\omega \cap \text{Ord} = \omega$$

by Corollary 4.3.5. Therefore, $\Box x \in \alpha$ implies $x \in \alpha$ by the last conjunct in Proposition 4.4.4.

On the other hand, assume that $x \in \alpha$ holds. Since $\alpha \subseteq \omega$, we have $x \in \omega = L_\omega \cap \text{Ord}$ by Corollary 4.3.5 while the modal postulates give $\Box x \in L_\beta$. Here we know $\beta \in \bar{\omega}_{(u)} \subseteq \bar{\omega}_{(\omega)}$, and therefore Corollary 4.4.8 would imply that $x \in L_\beta$ indeed.

Likewise, our assumption on φ implies that there is a unique $y \in L_\omega$ satisfying $\varphi(x, y)$, while the modal postulates give

$$\Box (\exists y' \in L_\beta \varphi(x, y')).$$

Observe that $\exists y' \in L_\beta \varphi(x, y')$ would imply $y \in L_\beta$ by uniqueness of y , thus we instead have $\Box y \in L_\beta$ here. Corollary 4.4.8 implies $y \in L_\beta$, so we have verified $x \in \{x \in L_\beta \cap \text{Ord} : \exists y \in L_\beta \varphi(x, y)\}$.

For the backward direction, by Lemma 4.4.2, $\bar{\omega}_{(u)} \subseteq \bar{\omega}_{(\omega)}$, thus $L_{\bar{\omega}_{(u)}} \subseteq L_{\bar{\omega}_{(\omega)}}$ and it suffices to show that

$$V(\mathcal{A}) \models \left\{ x \in L_{\bar{\omega}_{(\omega)}} : L_{\bar{\omega}_{(\omega)}} \models \Psi(x) \right\} \subseteq \bar{\omega}_{(\omega)}.$$

Take any $x \in L_{\bar{\omega}_{(\omega)}}$ and assume that $\Psi(x)$ holds. This means $x \in \text{def}(L_\alpha)$ for some $\alpha \in \bar{\omega}_{(\omega)}$. It follows by Proposition 4.4.4 that $\Box \alpha \in \omega$, and thus $\Box x \in L_\omega \cap \text{Ord}$, i.e. $\Box x \in \omega$ by Corollary 4.3.5. Additionally, we know that $\alpha \subseteq \omega$, thus

$$x \subseteq L_\alpha \cap \text{Ord} \subseteq L_\omega \cap \text{Ord} = \omega.$$

We have shown that $x \subseteq \omega \wedge \Box x \in \omega$. Combined with the second conjunct of $\Psi(x)$, we immediately know $x \in \bar{\omega}_{(\omega)}$ by Corollary 4.4.6. $\square$

Take any $u \subsetneq \omega$ that encompasses ω , then Lemma 4.4.2 implies that $V(\mathcal{A}) \models \bar{\omega}_{(u)} \subsetneq \bar{\omega}_{(\omega)}$. By Theorem 4.4.10, however, we have $\bar{\omega}_{(\omega)} \subseteq L_{\bar{\omega}_{(u)}}$. Therefore, it follows immediately that:

Corollary 4.4.11. *$V(\mathcal{A})$ realises $\exists \alpha \in \text{Ord} \neg L_\alpha \cap \text{Ord} = \alpha$.*

We finally show that additionally, $L_{\bar{\omega}_{(u)}} = L_{\bar{\omega}_{(\omega)}}$. We need another technical lemma:

Lemma 4.4.12 (IKP). *For any $\alpha, \beta \in \text{Ord}$ such that $\alpha, \beta \subseteq \omega$, if $\alpha \in \text{def}(L_\beta)$, then $L_\alpha \in \text{def}(L_\beta)$.*

Proof. Given such α and β , first observe that using the process described in the proof sketch of Theorem 4.3.4, it is not hard to show that for any $n \in \omega$, L_n is finite and thus $L_{n+} = \text{def}(L_n) = \mathcal{P}(L_n) \cap L_\omega$, and this characterisation is downwards absolute to L_β .

For any $n \in \alpha$, $n \in \text{def}(L_m) \subseteq L_\beta$ for some $m \in \beta$. It follows that $n \subseteq L_m \cap \text{Ord} = m$ by Corollary 4.3.5, thus $n \in m$ or $n = m$ by trichotomy. Either way, we have $L_n \in \text{def}(L_m) \subseteq L_\beta$. Additionally, $\text{def}(L_n) \subseteq \text{def}(L_m) \subseteq L_\beta$, thus $\text{def}(L_n) = \mathcal{P}(L_n) \cap L_\beta$ through our observation above.

Now, we just need to construct

$$L_\alpha = \{x \in L_\beta : \exists n \in L_\beta (n \in \alpha \wedge x \subseteq L_n)\} \in \text{def}(L_\beta),$$

and it suffices to check that there is a first-order formula $\psi(n, x)$ such that

$$(L_\beta \models \psi(n, x)) \leftrightarrow x = L_n$$

for any $n \in L_\beta \cap \text{Ord} \subseteq \omega$, $x \in L_\beta$.

We can easily characterise L_n by asserting that there exists a function f on domain n^+ such that $f(\emptyset) = \emptyset$ and $f(m^+) = \mathcal{P}(f(m)) \cap L_\beta$ for every $m \in n$. However, we need to take into account that the depth of Kuratowski pairs only guarantees the existence of such f corresponding, informally, to $n - 3$ inside $\text{def}(L_m)$ for any $m \geq n$. Therefore, let $\psi'(n, x)$ be the statement that such f exists on domain n^+ with $x = f(n)$, then we need to take $\psi(n, x)$ as

$$(n = 0 \wedge x = \emptyset) \vee (n = 1 \wedge x = \mathcal{P}(\emptyset)) \vee (n = 2 \wedge x = \mathcal{P}\mathcal{P}(\emptyset)) \\ \vee \exists m \in n \exists y \in x (\psi'(m, y) \wedge n = m^{+++} \wedge x = \mathcal{P}\mathcal{P}\mathcal{P}(y)). \quad \square$$

Corollary 4.4.13. *For any $u \subseteq \omega$ that encompasses ω through some Δ_0 -formula, we have*

$$V(\mathcal{A}) \models L_{\bar{\omega}(u)} = L_{\bar{\omega}(\omega)}.$$

It follows that $V(\mathcal{A})$ realises $\exists \alpha, \beta \in \text{Ord} (\alpha \neq \beta \wedge L_\alpha = L_\beta)$.

Proof. Working inside $V(\mathcal{A})$, by Lemma 4.4.2, we have $\bar{\omega}(u) \subseteq \bar{\omega}(\omega)$, thus $L_{\bar{\omega}(u)} \subseteq L_{\bar{\omega}(\omega)}$. So we just need to check the opposite direction.

Take any $\alpha \in \bar{\omega}(\omega)$, then by Theorem 4.4.10, there must be some $\beta \in \bar{\omega}(u)$ such that $\alpha \in \text{def}(L_\beta)$. We also know that $\alpha, \beta \subseteq \omega$ by Proposition 4.4.4, thus we know from Lemma 4.4.12 that $L_\alpha \in \text{def}(L_\beta)$. It follows that every set $x \in \text{def}(L_\alpha)$ will be of the form

$$x = \{y \in L_\alpha : L_\alpha \models \psi(y)\} \\ = \left\{y \in L_\beta : L_\beta \models y \in L_\alpha \wedge \psi(y)^{L_\alpha}\right\} \in \text{def}(L_\beta) \subseteq L_{\bar{\omega}(u)}.$$

It follows immediately that

$$L_{\bar{\omega}(\omega)} = \bigcup_{\alpha \in \bar{\omega}(\omega)} \text{def}(L_\alpha) \subseteq L_{\bar{\omega}(u)}. \quad \square$$

Chapter 5

Incomparable codings

We will now elaborate on general properties and operations on incomparable ordinals in intuitionistic set theory, and present the concrete mechanism I call ‘incomparable codings’ that encodes information from the internal structure of an arbitrary set into complex intuitionistic ordinals α and thus renders highly ‘non-constructible’ sets to be definable from α .

We shall say that two sets x and y are *weakly incomparable*, denoted $x \perp y$, if

$$\neg x \in y^+ \wedge \neg y \in x^+,$$

or equivalently $x \not\subseteq y \wedge x \neq y \wedge y \not\subseteq x$. Classically, there may be incomparable sets but never incomparable ordinals. However, in intuitionistic models, for example, we see the ordinals α and β used in the proof of Lemma 4.3.1 are easily realised to be weakly incomparable, thus CZF is consistent with the existence of weakly incomparable ordinals.

We say that two sets x and y are *strongly incomparable* if

$$\neg x \subseteq y \wedge \neg y \subseteq x.$$

Using Lemma 4.4.2, we see that the realisability model in the previous chapter gives

$$V(\mathcal{A}) \models \bar{\omega}_{(u)}, \bar{\omega}_{(v)} \text{ are strongly incomparable ordinals}$$

if and only if $u, v \subseteq \omega$ are strongly incomparable sets in the base theory. We have plenty of such sets, for example, $u = \omega \setminus \{1\}$ and $v = \omega \setminus \{2\}$, thus CZF is also consistent with the existence of strongly incomparable ordinals.

General theory on such ordinals will be developed in section 5.1, which will be

utilised to implement a specific realisability model of $\text{CZF} + V = L$ through the rest of the sections.

5.1 Incomparable ordinals and L

The key focus of our endeavour will be to build and use large collections of pairwise incomparable ordinals, also known as antichains. Classically, given an arbitrary set s , an antichain of size s could usually be given through a function f on domain s such that

$$\forall x, y \in s (\neg x = y \rightarrow f(x) \perp f(y)),$$

so that the range of f would be an antichain of the appropriate size.

However, observe that this definition is intuitionistically weak, essentially because both sides of the implication are negated statements. Thus, from some assumptions such as $f(x) = f(y)$, we can derive $\neg \neg x = y$ by contraposition but, in intuitionistic logic which lacks double negation elimination, cannot eventually derive $x = y$. Thus, we must rephrase the definition of an incomparable sequence to be in positive terms whenever possible.

Formally, we shall thus say that a function f on domain s is *pairwise weakly incomparable* if

$$\forall x, y \in s (\neg f(x) \in f(y) \wedge (f(x) = f(y) \rightarrow x = y)).$$

It is easy to see that this implies the classical definition above, whereas the converse implication would only hold in classical logic.

Likewise, we say that a function f on domain s is *pairwise strongly incomparable* if

$$\forall x, y \in s (f(x) \subseteq f(y) \rightarrow x = y).$$

Theory of weak incomparables

Important properties of weak incomparability are explored in my manuscript [47], which has been accepted for publication by the Journal of Symbolic Logic by the time of writing this thesis. There, an internal mechanism to systematically generate large amounts of weakly incomparable ordinals is demonstrated through ordinal arithmetic.

Firstly, we define *ordinal addition* intuitionistically as

$$\alpha + \beta = \alpha \cup \bigcup_{\gamma \in \beta} (\alpha + \gamma)^+.$$

This is the convention used in, for example, [3, Chapter 9]. As intuitionistic logic does not prove the trichotomy that every ordinal is either zero, a successor or a limit, we cannot define addition for the three cases separately.

Observe that:

Lemma 5.1.1 ([47, Lemma 5]). *For any $\alpha, \beta, \gamma \in \text{Ord}$, we have $\alpha + \beta \in \alpha + \gamma \rightarrow \beta \in \gamma$ and $\alpha + \beta = \alpha + \gamma \rightarrow \beta = \gamma$.*

Proof. We prove the conjunction of the two desired properties by simultaneous set induction on the parameters β and γ . First suppose that

$$\alpha + \beta \in \alpha + \gamma = \alpha \cup \bigcup_{\delta \in \gamma} (\alpha + \delta)^+.$$

Observe that if $\alpha + \beta \in \alpha$, then also $\alpha + \beta \in \alpha + \beta$ which contradicts the set induction axiom. Thus we must have $\alpha + \beta \in (\alpha + \delta)^+$ for some $\delta \in \gamma$, i.e. $\alpha + \beta \in \alpha + \delta$ or $\alpha + \beta = \alpha + \delta$. By the inductive hypothesis, we have $\beta \in \delta$ or $\beta = \delta$ and hence $\beta \in \gamma$ either way.

On the other hand, suppose that $\alpha + \beta = \alpha + \gamma$, then

$$\alpha + \delta \in (\alpha + \delta)^+ \subseteq \alpha + \beta = \alpha + \gamma$$

for any $\delta \in \beta$. By the inductive hypothesis, we have $\delta \in \gamma$ for any $\delta \in \beta$, i.e. $\beta \subseteq \gamma$. An entirely symmetrical argument shows that $\gamma \subseteq \beta$ as well, hence $\beta = \gamma$ as desired. $\square$

Lemma 5.1.2 ([47, Lemma 6]). *Suppose $\alpha \perp \beta \in \text{Ord}$, for any $\gamma \in \text{Ord}$, also $\alpha \perp \beta^+ + \gamma$.*

Proof. Firstly, suppose that $\beta^+ + \gamma \in \alpha^+$. Then $\beta \in \beta^+ + \gamma \subseteq \alpha$, contradicting our assumption that $\alpha \perp \beta$. Observe that this already implies $\neg \alpha = \beta^+ + \gamma$ for any $\gamma \in \text{Ord}$.

It suffices to further show by set induction on γ that $\neg \alpha \in \beta^+ + \gamma$. Suppose that $\alpha \in \beta^+ + \gamma$ yet $\neg \alpha \in \beta^+ + \delta$ for every $\delta \in \gamma$. Since also $\neg \alpha = \beta^+ + \delta$, i.e.

we have $\neg\alpha \in (\beta^+ + \delta)^+$, thus we must have $\alpha \in \beta^+$. This again contradicts $\alpha \perp \beta$. $\square$

Corollary 5.1.3 ([47, Corollary 7]). *Suppose $\alpha \perp \beta \in \text{Ord}$, then for any $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \text{Ord}$, we have*

$$(\alpha^+ + \gamma_1) \cup (\beta^+ + \gamma_2) = (\alpha^+ + \delta_1) \cup (\beta^+ + \delta_2) \rightarrow \gamma_1 = \delta_1 \wedge \gamma_2 = \delta_2.$$

Proof. Assume $(\alpha^+ + \gamma_1) \cup (\beta^+ + \gamma_2) = (\alpha^+ + \delta_1) \cup (\beta^+ + \delta_2)$. For any $\eta \in \gamma_1$, we have $\alpha^+ + \eta \in \alpha^+ + \gamma_1 \subseteq (\alpha^+ + \delta_1) \cup (\beta^+ + \delta_2)$. Suppose that $\alpha^+ + \eta \in \beta^+ + \delta_2$, then $\alpha \in \beta^+ + \delta_2$ by transitivity, and this contradicts Lemma 5.1.2. This means we must have $\alpha^+ + \eta \in \alpha^+ + \delta_1$ and thus $\eta \in \delta_1$ by Lemma 5.1.1, i.e. $\gamma_1 \subseteq \delta_1$. Entirely symmetrical arguments show that $\delta_1 \subseteq \gamma_1$, $\gamma_2 \subseteq \delta_2$ and $\delta_2 \subseteq \gamma_2$ as well, hence $\gamma_1 = \delta_1$ and $\gamma_2 = \delta_2$ as desired. $\square$

Corollary 5.1.3 essentially claims that given two weakly incomparable ordinals, there is a method to represent any pair of two ordinals as a single ordinal uniformly. We proceed to see that this generates an encoding that is not only injective, but also (pairwise) weakly incomparable:

Proposition 5.1.4 ([47, Proposition 8]). *Suppose $\alpha \perp \beta \in \text{Ord}$, for any set $s \subseteq \text{Ord}$ of ordinals, the function $f : s \rightarrow \text{Ord}$ given by*

$$f(\gamma) = \alpha^+ \cup (\beta^+ + \gamma)$$

is pairwise weakly incomparable.

Proof. We first show that $\neg f(\gamma) \in f(\delta)$ for any $\gamma, \delta \in s$. Suppose that $\alpha^+ \cup (\beta^+ + \gamma) \in \alpha^+ \cup (\beta^+ + \delta)$. Notice that $\alpha^+ \cup (\beta^+ + \gamma) \in \alpha^+$ implies $\alpha \in \alpha^+ \cup (\beta^+ + \gamma) \subseteq \alpha$, contradicting set induction, thus we must have $\alpha^+ \cup (\beta^+ + \gamma) \in \beta^+ + \delta$ instead. However, by Lemma 5.1.2, we have $\neg\alpha \in \beta^+ + \delta$ and hence another contradiction. Thus we must have $\neg\alpha^+ \cup (\beta^+ + \gamma) \in \alpha^+ \cup (\beta^+ + \delta)$.

Now, it remains to show that

$$\alpha^+ \cup (\beta^+ + \gamma) = \alpha^+ \cup (\beta^+ + \delta) \rightarrow \gamma = \delta.$$

This is simply an instance of Corollary 5.1.3 where $\gamma_1 = \delta_1 = \emptyset$. $\square$

Given fixed $\alpha \perp \beta \in \text{Ord}$ and any function $f : x \rightarrow \text{Ord}$, we shall define

$f^\perp : x \rightarrow \text{Ord}$ by

$$f^\perp(y) = \alpha^+ \cup (\beta^+ + f(y)).$$

By Proposition 5.1.4, $f^\perp$ is pairwise weakly incomparable if and only if f is injective, i.e. f satisfies $\forall y, z \in x (f(y) = f(z) \rightarrow y = z)$. In [47], I continued by iterating similar constructions and derive that:

Theorem 5.1.5 (IKP). *Suppose $\alpha \perp \beta \in \text{Ord}$, then there is a Σ_1 -class function F such that for any set x , $F(x) : x \rightarrow \text{Ord}$ is pairwise weakly incomparable.*

The precise constructions are quite technical and are omitted in this thesis.

Now, incomparable ordinal functions are important because they allow us to encode membership information for subsets of arbitrarily complex sets through ordinal constructions. This idea is briefly commented on by Matthews and Rathjen in the final section of [25], and can be made precise in the following way:

Proposition 5.1.6. *Suppose that $f : x \rightarrow \text{Ord}$ is pairwise weakly incomparable and $f \in L$, then for any $u \subseteq x$, there exists a corresponding ordinal δ_u such that $\delta_u \in L$ implies $u \in L$.*

Proof. One simply takes $\delta_u = \bigcup_{y \in u} f(y)^+$, which is an ordinal trivially. Let

$$v = \{y \in x : f(y) \in \delta_u\},$$

then $u \subseteq v$ trivially. Observe that for any $y \in v$, we must have $z \in u$ such that $f(y) \in f(z)^+$ by definition of δ_u . However, weak incomparability means that $\neg f(y) \in f(z)$ and that $f(y) = f(z)$ must imply $y = z$. Therefore, we have $y = z \in u$ and thus $v = u$ as a consequence.

It suffices to observe that $f, x \in L$ (since x is the domain of f). Thus if also $\delta_u \in L$, then v is a definable set from parameters in L and thus $u = v \in L$. $\square$

Using this construction, we can derive the main result in [47]:

Theorem 5.1.7 (IKP, [47, Theorem 2]). *Assume there exist $\alpha \perp \beta \in \text{Ord}$, then the following are equivalent:*

- (i) $\text{Ord} \subseteq L$,
- (ii) $V = L$.

Proof. The backward direction is trivial. For the forward direction, we use set induction on x to show that $\forall x x \in L$.

From the inductive hypothesis, we will know that $x \subseteq L$, so by strong Σ_1 -collection (which is a theorem in IKP, cf. [3, Theorem 19.1.5]) we have a set $s \subseteq \text{Ord}$ such that for all $y \in x$, there exists $\gamma \in s$ such that $y \in L_\gamma$. Take $\sigma = \bigcup_{\gamma \in s} \gamma$, then $x \subseteq L_\sigma$.

Now, we use the crucial fact that the construction of the Σ_1 -class function F in Theorem 5.1.5 uses only the weak theory IKP, and it is verified in [25, section 5] that for every theorem φ of IKP, $\text{IKP} \vdash \varphi^L$. It follows that we can compute $F^L(L_\sigma) \in L$ and, since Σ_1 -statements are upwards absolute, we must have

$$F(L_\sigma) = F^L(L_\sigma) \in L.$$

It follows from Proposition 5.1.6 that there exists an ordinal δ_x such that $\delta_x \in L$ implies $x \in L$. Since we assumed $\text{Ord} \subseteq L$, this must be the case. $\square$

Corollary 5.1.8 ([47, Corollary 14]³⁷). *Let theory T be any extension of IKP that proves the powerset $\mathcal{P}(1)$ exists. If $T + V \neq L + 2 \neq \mathcal{P}(1)$ is consistent, then $T \not\vdash \text{Ord} \subseteq L$.*

Proof. Working inside the theory T , it is easy to verify that $2 = \{0, 1\}$ and $\mathcal{P}(1)$ are both ordinals, and clearly $2 \notin \mathcal{P}(1)$ and $\mathcal{P}(1) \notin 2$. Therefore,

$$T + 2 \neq \mathcal{P}(1) \vdash 2 \perp \mathcal{P}(1).$$

Suppose that $T \vdash \text{Ord} \subseteq L$, then by Theorem 5.1.7, we can derive $V = L$ inside $T + 2 \neq \mathcal{P}(1)$. This contradicts the assumption that $T + V \neq L + 2 \neq \mathcal{P}(1)$ is consistent. $\square$

Observe here that, if we work with the strong theory $T = \text{IZF}$, then by its full separation scheme, for any formula φ which does not contain the variable x , we can consider some ordinal

$$\alpha_\varphi = \{x \in 1 : \varphi\} \in \mathcal{P}(1),$$

thus $2 = \mathcal{P}(1)$ would imply $\alpha_\varphi = 0 \vee \alpha_\varphi = 1$ and thus the law of excluded middle, $\varphi \vee \neg\varphi$. Therefore, $\text{IZF} + 2 = \mathcal{P}(1)$ is simply equivalent to classical ZF over intuitionistic logic.

³⁷This is a simplified variant of the original statement in the published manuscript. The simplification is proposed by Jem Lord through private communication.

Informally, we can call some statement ψ an *anticlassical axiom*³⁸ if ZF (or any reasonable classical base theory) proves $\neg\psi$, yet ψ is consistent with some intuitionistic theory such as IZF. It follows from the previous remark that, for any anticlassical axiom ψ , $\text{IZF} + 2 = \mathcal{P}(1) \vdash \neg\psi$ over intuitionistic logic. By contraposition,

$$\text{IZF} + \psi \vdash 2 \neq \mathcal{P}(1).$$

Therefore, if for any such axiom ψ , $\text{IZF} + V \neq L + \psi$ is consistent, then the presumptions in Corollary 5.1.8 are obtained and we have $\text{IZF} \not\vdash \text{Ord} \subseteq L$.

Analogously, when we consider the Δ_0 -instances of separation in the weaker theory $\text{CZF}(\mathcal{P})$, we see that if any model of $\text{CZF}(\mathcal{P}) + V \neq L$ also violates some Δ_0 -instance of excluded middle, for example, if

$$\text{CZF}(\mathcal{P}) + V \neq L + \neg\forall x, y (x \in y \vee x \notin y)$$

is consistent, then $\text{CZF}(\mathcal{P}) \not\vdash \text{Ord} \subseteq L$.

Unfortunately, however, I know of no such interpretations, and thus Problem 4.3.3 remains open. The main difficulty is precisely as we show in this chapter, that the presence of specific coding objects in L can cause it to be much larger than what we traditionally conceive in classical set theory, thus making the implementation or verification of $V \neq L$ very complicated in intuitionistic models.

Theory of strong incomparables

Now, Theorem 5.1.7 is a surprising result in itself, since it relates in intuitionistic set theory the statement $\text{Ord} \subseteq L$, which is completely trivial in classical set theory, with $V = L$, which is easily demolished classically through forcing. However, working with weak incomparables can be unsatisfying since whether $\text{Ord} \subseteq L$ remains an open question in intuitionistic set theory (cf. section 4.3) and thus, constructions in Proposition 5.1.6 do not produce ordinal codes that lie readily in L .

If the ordinals in the range of the incomparable function are particularly nice, then there are analogous constructions which guarantee membership in L . For example, suppose that $f : x \rightarrow \text{Ord}$ has all of its values given by some

³⁸There are many well-known anticlassical axioms — for example, Church’s thesis, the uniformity principle, etc. We will not present any details here, but refer the readers to Chen and Rathjen [9] for elaboration, with explicit realisability constructions showing that they are consistent with IZF.

$g : x \rightarrow \text{Ord}$ as

$$\forall y \in x \ f(y) = L_{g(y)} \cap \text{Ord},$$

then for any $u \subseteq x$, we can construct

$$\delta_u^* = \bigcup_{y \in u} \text{def}(L_{g(y)}) \cap \text{Ord} = L_{\bigcup_{y \in u} g(y)^+} \cap \text{Ord} \in L.$$

We can prove an analogue of Proposition 5.1.6 using the ordinals δ_u^* . Observe carefully that now, it is necessary to assume that the function is pairwise strongly incomparable:

Proposition 5.1.9. *Suppose that $f : x \rightarrow \text{Ord}$ is pairwise strongly incomparable, $f \in L$, and f is given by some $g : x \rightarrow \text{Ord}$ as $f(y) = L_{g(y)} \cap \text{Ord}$ for every $y \in x$. Then for any $u \subseteq x$, we have $u \in L$.*

Proof. We know that f, x and δ_u^* as constructed above are in L . Let

$$v = \{y \in x : f(y) \in \delta_u^*\},$$

then $v \in L$. We likewise trivially have $u \subseteq v$. For any $y \in v$, we must have $z \in u$ such that $f(y) \in \text{def}(L_{g(z)}) \cap \text{Ord}$ by definition of δ_u^* . This implies that $f(y) \subseteq L_{g(z)} \cap \text{Ord} = f(z)$, and strong incomparability implies that $y = z$. Therefore, we have $y = z \in u$ and thus $u = v \in L$ as desired. $\square$

Specifically, recall that Corollary 4.3.5 states $\forall n \in \omega \ L_n \cap \text{Ord} = n$. Thus, for any $\alpha \in \text{Ord}$ such that $\alpha \subseteq \omega$, we immediately have

$$\alpha = \bigcup_{n \in \alpha} n^+ = \bigcup_{n \in \alpha} \text{def}(L_n) \cap \text{Ord} = L_{\bigcup_{n \in \alpha} n^+} \cap \text{Ord} = L_\alpha \cap \text{Ord}.$$

It follows immediately that:

Corollary 5.1.10. *Suppose that $f : x \rightarrow \mathcal{P}(\omega) \cap \text{Ord}$ is pairwise strongly incomparable and $f \in L$, then for any $u \subseteq x$, we have $u \in L$.*

However, given these additional requirements, it is no longer possible to automatically generate large collections of incomparables using methods similar to Proposition 5.1.4. Later in section 5.4, we will make use of recursion-theoretic constructions to manually build a pairwise strongly incomparable function $\omega \rightarrow \mathcal{P}(\omega) \cap \text{Ord}$ in L ; in the next chapter, we will then develop some semi-automatic mechanisms to construct larger sets of strongly incomparable ordinals given a base theory containing the powerset axiom.

5.2 The case for $V = L$

In the remainder of this chapter, we shall apply the method of incomparable codings to analyse the strength implications of the axiom $V = L$, following the constructions in my arXiv manuscript [49]. Thus, let us first understand why it is interesting — and non-trivial — to study $V = L$ for certain set theories. We reproduce the introduction to [49] here:

Traditionally, most set theories T we work with admit the theorem

$$T \vdash T^L,$$

and it follows immediately that the ‘inner model’ L is a model of the extension $T + V = L$, thus T and $T + V = L$ are equiconsistent. This is the case for well-known classical theories KP, ZF and ZFC; as we know from Lubarsky [24] and Matthews and Rathjen [25], this is also the case for intuitionistic theories IKP and IZF.

However, there are known theories T that do not satisfy $T \vdash T^L$ and violate the equiconsistency claim. For example, Rathjen proved that:

Theorem 5.2.1 ([36, Theorem 4.6]). *KP($\mathcal{P}$) + $V = L$ is much stronger than KP($\mathcal{P}$), where KP($\mathcal{P}$) is classical Kripke–Platek set theory with a new primitive unary function symbol $\mathcal{P}$ for powerset (see remarks in section 4.1 and footnote 29).*

The key step in this proof is the following lemma:

Lemma 5.2.2. *KP($\mathcal{P}$) + ω_1 exists proves the consistency of KP($\mathcal{P}$).*

Proof sketch. Using the technique in [30, section 4], we can construct the Bachmann–Howard ordinal $\vartheta_{\varepsilon_{\Omega+1}}(0)$ as some class of ordinals Acc in KP($\mathcal{P}$). Assuming ω_1 exists, Acc is a Σ_1 -definable subclass of ω_1 whose existential quantifier can be easily bounded to range over sets of functions on ω_1 , thus Acc is actually a subset of ω_1 and $\vartheta_{\varepsilon_{\Omega+1}}(0)$ exists as a set-theoretic ordinal.

One then applies the method of relativised ordinal analysis in [35]. By [35, Theorem 8.2], if KP($\mathcal{P}$) proves a Σ_1 -sentence φ (possibly containing the powerset symbol), then

$$V_{\vartheta_{\varepsilon_{\Omega+1}}(0)} \models \varphi.$$

Therefore, Con(KP($\mathcal{P}$)) follows immediately. □

Proof of Theorem 5.2.1. It is easy to verify in classical set theory that for any countable ordinal $\alpha \in \text{Ord}$, L_α is also countable. Thus, assuming $V = L$, then $\mathcal{P}(\omega) \in L$ is uncountable, and we must have $\mathcal{P}(\omega) \in L_\alpha$ for some uncountable $\alpha \in \text{Ord}$. It follows that ω_1 exists, and by Lemma 5.2.2, $\text{KP}(\mathcal{P}) + V = L$ proves the consistency of $\text{KP}(\mathcal{P})$. $\square$

On the intuitionistic side, we similarly know that:

Proposition 5.2.3. $\text{CZF}(\mathcal{P}) + \neg\text{Exp}^L$ is consistent. Since $\text{CZF} \vdash \text{Exp}$ by Proposition 4.1.2, $\text{CZF}(\mathcal{P}) \not\vdash \text{CZF}^L$.

Proof. Let REM denote the axiom scheme of *restricted excluded middle*, i.e. $\varphi \vee \neg\varphi$ for any Δ_0 -formula φ . Observe that $\text{CZF} + \text{REM}$ suffices for all uses of the classical logic in Theorem 5.2.1, namely that the ordinals are linearly ordered and L_α is countable for any countable $\alpha \in \text{Ord}$. Therefore, inside³⁹ $\text{CZF}(\mathcal{P}) + \text{REM} + V = L$, we similarly consider $\mathcal{P}(\omega) \in L_\alpha$ and α must then contain all countable ordinals; so $\text{CZF}(\mathcal{P}) + \text{REM} + V = L$ also proves that the Bachmann–Howard ordinal $\vartheta_{\varepsilon_{\Omega+1}}(0)$ exists.

We know that the relativised ordinal analysis can be performed in an intuitionistic base theory (cf. remarks in [38, section 3(c)]). On the other hand, $\text{CZF}(\mathcal{P}) + \text{REM}$ is equiconsistent with $\text{KP}(\mathcal{P})$, using the same weak realisability argument we mentioned in section 4.1 and will explain in more detail in section 6.2. Therefore, it shall follow that

$$\text{CZF}(\mathcal{P}) + \text{REM} + V = L \vdash \text{Con}(\text{CZF}(\mathcal{P}) + \text{REM}).$$

Now, suppose that $\text{CZF}(\mathcal{P}) + \neg\text{Exp}^L$ is not consistent, i.e. $\text{CZF}(\mathcal{P}) \vdash \neg\neg\text{Exp}^L$. It follows that $\text{CZF}(\mathcal{P}) + \text{REM} \vdash \neg\neg\text{Exp}^L$. However, the axioms of exponentiation, subset collection and powerset are the sole obstacles preventing L from being a model of $\text{CZF}(\mathcal{P})$ here, as extending the proof of $L \models \text{IKP}$ in [25, section 5], it is easy to verify that $\text{CZF}(\mathcal{P}) + \text{REM}$ proves

$$L \models \text{CZF} - \text{subset collection} + \text{REM} + V = L.$$

Therefore, combining with $\text{CZF}(\mathcal{P}) + \text{REM} \vdash \neg\neg\text{Exp}^L$ and the fact that REM and exponentiation together imply the powerset axiom, we have $\text{CZF}(\mathcal{P}) +$

³⁹REM also suffices for deriving the equivalence between the axioms of exponentiation, subset collection and powerset in CZF (cf. Proposition 4.1.2 or [3, section 10.1]), thus $\text{CZF}(\mathcal{P}) + \text{REM}$ and $\text{CZF} + \text{REM}$ are actually the same theory. However, we will continue to write $\text{CZF}(\mathcal{P}) + \text{REM}$ for clarity reasons.

$\text{REM} \vdash \neg\neg\varphi^L$ for every axiom φ of $\text{CZF}(\mathcal{P}) + \text{REM} + V = L$. It follows that, if there is a proof of $\text{CZF}(\mathcal{P}) + \text{REM} + V = L \vdash \perp$, then there is also a proof of $\text{CZF}(\mathcal{P}) + \text{REM} \vdash \neg\neg\perp^L$, which is simply $\text{CZF}(\mathcal{P}) + \text{REM} \vdash \perp$ since intuitionistic logic admits triple negation elimination.

In other words, $\text{Con}(\text{CZF}(\mathcal{P}) + \text{REM})$ implies $\text{Con}(\text{CZF}(\mathcal{P}) + \text{REM} + V = L)$, and both are proved in $\text{CZF}(\mathcal{P}) + \text{REM} + V = L$ by the argument above. This is a contradiction to Gödel's second incompleteness theorem. $\square$

This is a slightly stronger result than [25, Theorem 7.12], where a variant of weak realisability is used to directly show $\text{CZF}(\mathcal{P}) \not\vdash \text{Exp}^L$. Either way, we know that for any fragment $T \subseteq \text{CZF}(\mathcal{P})$ that proves Exp , T cannot prove T^L . This leaves open the question whether we still have $T \equiv_{\text{Con}} T + V = L$ for such theories, most notably when $T = \text{IKP}(\mathcal{P})$, CZF and $\text{CZF}(\mathcal{P})$. In what follows, we shall resolve the case for CZF and prove that, despite the proposition above, we still have

$$\text{CZF} \equiv_{\text{Con}} \text{CZF} + V = L.$$

The case for $\text{IKP}(\mathcal{P})$ and $\text{CZF}(\mathcal{P})$ will then be treated in chapter 6.

5.3 A universe over Martin-Löf type theory

We will construct a realisability interpretation of $\text{CZF} + V = L$ following the structure of Aczel's interpretation of CZF in Martin-Löf type theory with a single W -type (sometimes known as ML_1V) in [2]. Roughly speaking, the set-theoretic universe is defined by the type-theoretic constructor

$$A : \text{Type}, f : A \rightarrow V \vdash \text{sup}(A, f) : V,$$

where A ranges over the 'small' types in the theory.

As a basis, we use a presentation of the type theory given in [34]. There, it is implemented as a definable structure over the natural numbers using inductive definitions in the classical theory BI of second-order arithmetic. Since we have already presented Σ -inductive definitions for IKP in Theorem 4.2.1, we would conveniently perform the same constructions inside classical Kripke-Platek set theory KP instead.

We fix the pca Kl , Kleene's first algebra, which is clearly a Δ_0 -definable partial structure over ω in KP . We shall implement the semantics of the type theory

by interpreting the function types as collections of elements (with appropriate domains) in KL. In the type theory, we shall have the following types⁴⁰:

- (1) for each $n \in \omega$, $\bar{n}$ names a finite type with n elements;
- (2) $\bar{\omega}$ names the type of natural numbers;
- (3) $\bar{X}$ names a new, arbitrary primitive type that will later be specified through its interpretation in the base theory — this is our addition to ML_1V in order to introduce incomparable codings into the interpretation of CZF;
- (4) the usual type formers σ, π that build dependent Σ - and Π -types.

All type names are encoded as natural numbers in the following way:

$$\begin{aligned}\bar{n} &= \mathbf{p}0n \quad \text{for each } n \in \omega, \\ \bar{\omega} &= \mathbf{p}10, & \sigma &= \lambda nm. \mathbf{p}2(\mathbf{p}nm), \\ \bar{X} &= \mathbf{p}11, & \pi &= \lambda nm. \mathbf{p}3(\mathbf{p}nm).\end{aligned}$$

Fix any set $A \subseteq \omega$, we first invoke Σ -inductive definitions to define the universe of small types, $\mathbf{U}_A = I(\Phi)$, where Φ is a Δ_0 -class given by the following clauses:

- (1) $\langle \langle \bar{n}, n \rangle, \emptyset \rangle \in \Phi$ for each $n \in \omega$, which immediately implies $\langle \bar{n}, n \rangle \in \mathbf{U}_A$;
- (2) $\langle \langle \bar{\omega}, \omega \rangle, \emptyset \rangle \in \Phi$, which implies $\langle \bar{\omega}, \omega \rangle \in \mathbf{U}_A$;
- (3) $\langle \langle \bar{X}, A \rangle, \emptyset \rangle \in \Phi$, which implies $\langle \bar{X}, A \rangle \in \mathbf{U}_A$;
- (4) for any $n, e \in \omega$, if $f : s \rightarrow \mathcal{P}(\omega)$ is a set-theoretic function on some domain $s \subseteq \omega$, such that $n \in s$ and for any $k \in f(n)$,

$$ek \downarrow \in s,$$

then $\langle \langle \sigma ne, p \rangle, f \rangle \in \Phi$ where

$$p = \{ \mathbf{p}ku : k \in f(n), u \in f(ek) \},$$

and also $\langle \langle \pi ne, q \rangle, f \rangle \in \Phi$ where

$$q = \{ d \in \omega : \forall k \in f(n) \, dk \downarrow \in f(ek) \}.$$

⁴⁰We follow the notation used in [49], which was taken from Rathjen [34]. The overlined type names here are unrelated to the similarly named objects in section 4.2.

The idea is that $\mathbf{U}_A$ shall be a partial function on ω , such that for any type name n , $\mathbf{U}_A(n)$ is the collection of all elements in that type, where elements in Π -types are represented by objects in the pca Kl. We check that the inductive definition ensures that $\mathbf{U}_A$ is a function:

Lemma 5.3.1 (KP). *For any $n \in \omega$, if $\langle n, p \rangle, \langle n, q \rangle \in \mathbf{U}_A$, then $p = q$.*

Proof. By condition (ii) in Theorem 4.2.1, it suffices to verify that

$$\forall \langle \langle n, p \rangle, y \rangle \in \Phi \ (\forall \langle m, p' \rangle \in y \ \forall \langle m, q \rangle \in \mathbf{U}_A \ p' = q \rightarrow \forall \langle n, q \rangle \in \mathbf{U}_A \ p = q).$$

This is clear since by Corollary 4.2.2, any $\langle n, q \rangle \in \mathbf{U}_A$ must correspond to some $\langle \langle n, q \rangle, y \rangle \in \Phi$ where $y \subseteq \mathbf{U}_A$. Looking at the definition of Φ , we see that the set p corresponding to $n \in \omega$ is uniquely determined by relevant information in y , and therefore, a different $\langle n, q \rangle$ cannot lie in $\mathbf{U}_A$ if no alternative values for types in y can be found. $\square$

Now, $\mathbf{U}_A$ is given as a Σ_1 -class, so we can use it to again inductively define the W -type in the theory, implemented as a class $\mathbf{V}_A = I(\Psi) \subseteq \omega$ in KP, using the Σ_1 -class $\Psi \subseteq \omega \times \mathcal{P}(\omega)$ given by

$$\langle n, y \rangle \in \Psi \quad :\Longleftrightarrow \quad n \in \omega \wedge \exists \langle m, p \rangle \in \mathbf{U}_A \ (\mathbf{p}_0 n \downarrow = m \wedge \forall k \in p \ \mathbf{p}_1 n k \downarrow \in y).$$

This enforces that each element in $\mathbf{V}_A$ should be of the form $\mathbf{p}ne$ where n lies in the domain of $\mathbf{U}_A$ and $e \in \text{Kl}$ maps each $k \in \mathbf{U}_A(n)$ to $ek \downarrow \in \mathbf{V}_A$.

$\mathbf{V}_A$ will serve as the domain of set names, on which we build an interpreted realisability model of CZF. Just as in section 4.2, we specify realisers for the atomic formulae by simultaneous recursion; however, since the names for sets are now given as objects in $\mathbf{V}_A \subseteq \text{Kl}$, it suffices to use Corollary 2.1.2 for the pca to build $f_\in, g_= \in \text{Kl}$ as

$$\begin{aligned} f_\in &= \lambda \alpha \beta. \sigma(\mathbf{p}_0 \beta)(\lambda x. g_= \alpha(\mathbf{p}_1 \beta x)), \\ g_= &= \lambda \alpha \beta. \sigma(\pi(\mathbf{p}_0 \alpha)(\lambda y. f_\in(\mathbf{p}_1 \alpha y) \beta))(\lambda x. \pi(\mathbf{p}_0 \beta)(\lambda y. f_\in(\mathbf{p}_1 \beta y) \alpha)). \end{aligned}$$

Using induction on $\mathbf{V}_A$ as given by condition (ii) in Theorem 4.2.1, it is easy to verify that $f_\in \alpha \beta$ and $g_= \alpha \beta$ lie in the domain of $\mathbf{U}_A$ for any $\alpha, \beta \in \mathbf{V}_A$, i.e. they are proper type names. We can thus define the atomic clauses of the realisability conditions here as

$$d \Vdash \alpha \in \beta \quad :\Longleftrightarrow \quad d \in \mathbf{U}_A(f_\in \alpha \beta),$$

$$d \Vdash \alpha = \beta \quad :\Longleftrightarrow \quad d \in \mathbf{U}_A(g_{=} \alpha \beta).$$

The conditions for bounded quantifiers are now

$$\begin{aligned} d \Vdash \forall x \in \alpha \varphi & \quad :\Longleftrightarrow \quad \forall i \in \mathbf{U}_A(\mathbf{p}_0 \alpha) \, di \downarrow \Vdash \varphi[\mathbf{p}_1 \alpha i / x], \\ d \Vdash \exists x \in \alpha \varphi & \quad :\Longleftrightarrow \quad \mathbf{p}_0 d \downarrow \in \mathbf{U}_A(\mathbf{p}_0 \alpha) \wedge \mathbf{p}_1 d \Vdash \varphi[\mathbf{p}_1 \alpha(\mathbf{p}_0 d) / x], \end{aligned}$$

so that it is again not hard to verify that

$$\begin{aligned} (d \Vdash \alpha \in \beta) & \leftrightarrow (d \Vdash \exists x \in \beta \, \alpha = x), \\ (d \Vdash \alpha = \beta) & \leftrightarrow (d \Vdash \alpha \subseteq \beta \wedge \beta \subseteq \alpha). \end{aligned}$$

Just like in section 4.2, we treat $\mathbf{V}_A$ as the name of this realisability model, and write $\mathbf{V}_A \models \varphi$ to abbreviate for

$$\exists d \in \mathbf{Kl} \, d \Vdash \varphi,$$

where φ is a first-order formula possibly containing parameters from $\mathbf{V}_A$.

Proceeding to the axioms, we see that the realisers for the equality axioms are entirely analogous to the constructions in section 4.2. The realisers for the bounded quantifier axioms need to be adjusted slightly to accommodate for the interpreted realisability conditions for unbounded quantifiers, but still use the equality axioms in essentially the same way as Lemma 4.2.5. The axioms of extensionality and set induction are also analogous.

For the pairing axiom, we will utilise the primitive type $\bar{2}$ in $\mathbf{U}_A$ and construct a function in $\mathbf{Kl}$ that maps any $\alpha, \beta \in \mathbf{V}_A$ to an object

$$\{\alpha, \beta\}^V = \mathbf{p}\bar{2}(\lambda x. \mathbf{d}_{\mathbb{N}} x 0 \alpha \beta) \in \mathbf{V}_A.$$

For the union axiom, we use a Σ -type and build

$$\bigcup^V \alpha = \mathbf{p}(\sigma(\mathbf{p}_0 \alpha)(\lambda x. \mathbf{p}_0(\mathbf{p}_1 \alpha x)))(\lambda x. \mathbf{p}_1(\mathbf{p}_1 \alpha(\mathbf{p}_0 x))(\mathbf{p}_1 x)) \in \mathbf{V}_A.$$

For infinity, we build a map $g \in \mathbf{Kl}$ from ω to $\mathbf{V}_A$ as

$$g = \mathbf{r}(\lambda a f. \mathbf{p}\bar{a} f) = \mathbf{r}(\lambda a f. \mathbf{p}(\mathbf{p}_0 a) f)$$

by recursion, so that $gn = \mathbf{p}\bar{n}(\lambda m. gm)$. Then the names for natural numbers

and ω can be given as

$$\begin{aligned} n^V &= gn \in \mathbf{V}_A \quad \text{for all } n \in \omega, \\ \omega^V &= \mathbf{p}\bar{\omega}g \in \mathbf{V}_A. \end{aligned}$$

Verifying that these constructions work is again analogous to section 4.2.

The realiser for strong collection is slightly different, as we observe that we do not need assumptions from the base theory for the collection scheme in an *interpreted* realisability model of set theory:

Lemma 5.3.2. *The strong collection scheme (6') is realised in $\mathbf{V}_A$:*

$$\forall x \in a \exists y \theta(x, y) \rightarrow \exists b (\forall x \in a \exists y \in b \theta(x, y) \wedge \forall y \in b \exists x \in a \theta(x, y)).$$

Proof. Let $\vec{z}$ enumerate the free variables in $\theta(x, y)$ other than x, y and a . For any $\alpha \in \mathbf{V}_A$, $\vec{\beta} \in \mathbf{V}_A^k$ and $f \Vdash \forall x \in a \exists y \theta(x, y) [\alpha/a, \vec{\beta}/\vec{z}]$, we can define

$$g = \lambda \alpha f. \mathbf{p}(\mathbf{p}_0 \alpha)(\lambda x. \mathbf{p}_0(fx)),$$

then for any $k \in \mathbf{U}_A(\mathbf{p}_0 \alpha)$, $\mathbf{p}_1(fk) \Vdash \theta[\alpha/a, \mathbf{p}_1 \alpha k/x, \mathbf{p}_1(g \alpha f)k/y, \vec{\beta}/\vec{z}]$. Thus

$$\lambda \alpha \vec{\beta} f. \mathbf{p}(g \alpha f)(\mathbf{p}(\lambda k. \mathbf{p}k(\mathbf{p}_1(fk)))(\lambda k. \mathbf{p}k(\mathbf{p}_1(fk))))$$

realises the axiom. □

For subset collection, we use a Π -type:

Lemma 5.3.3. *The subset collection scheme (8) is realised in $\mathbf{V}_A$:*

$$\begin{aligned} &\exists c \forall u (\forall x \in a \exists y \in b \psi(x, y, u) \\ &\rightarrow \exists d \in c (\forall x \in a \exists y \in d \psi(x, y, u) \wedge \forall y \in d \exists x \in a \psi(x, y, u))), \end{aligned}$$

Proof. Let $\vec{z}$ enumerate the free variables in $\psi(x, y, u)$ other than x, y, u, a, b and c . We define

$$c^* = \lambda \alpha \beta. \mathbf{p}(\pi \alpha(\lambda x. \beta))(\lambda h. \mathbf{p}(\mathbf{p}_0 \alpha)(\lambda x. \mathbf{p}_1 \beta(hx))),$$

such that for any $\alpha, \beta \in \mathbf{V}_A$, $c^* \alpha \beta \in \mathbf{V}_A$. Now, suppose that some

$$f \Vdash \forall x \in a \exists y \in b \psi(x, y, u) [\alpha/a, \beta/b, c^* \alpha \beta/c, \delta/u, \vec{\gamma}/\vec{z}],$$

and we will let

$$d^* = \mathbf{p}(\mathbf{p}_0\alpha)(\lambda x. \mathbf{p}_1\beta(\mathbf{p}_0(fx))) = \mathbf{p}_1(c^*\alpha\beta)(\lambda x. \mathbf{p}_0(fx)).$$

Then $\mathbf{p}(\lambda x. \mathbf{p}x(\mathbf{p}_1(fx)))(\lambda x. \mathbf{p}x(\mathbf{p}_1(fx)))$ realises the corresponding conjunction that d^* needs to satisfy as d . Therefore,

$$\lambda\alpha\beta\vec{\gamma}. \mathbf{p}(c^*\alpha\beta)(\lambda\delta f. \mathbf{p}(\lambda x. \mathbf{p}_0(fx))(\mathbf{p}(\lambda x. \mathbf{p}x(\mathbf{p}_1(fx)))(\lambda x. \mathbf{p}x(\mathbf{p}_1(fx)))))$$

realises the axiom. $\square$

For Δ_0 -separation, the usual argument depends on the observation that the collection of realisers for each Δ_0 -formula forms a type in $\mathbf{U}_A$. However, we use the same trick as in section 4.2 and just treat binary intersection instead, where we only need to use the type for realisers in the atomic case:

Lemma 5.3.4. *The binary intersection axiom is realised in $\mathbf{V}_A$:*

$$\forall a \forall b \exists y \forall x (x \in y \leftrightarrow x \in a \wedge x \in b).$$

Proof. Just as in Lemma 4.2.12, we consider

$$\gamma = \lambda\alpha\beta. \mathbf{p}(\sigma(\mathbf{p}_0\alpha)(\lambda x. f_{\in}(\mathbf{p}_1\alpha x)\beta))(\lambda x. \mathbf{p}_1\alpha(\mathbf{p}_0x)),$$

and we will have

$$(\mathbf{p}(\mathbf{p}fg)d \Vdash \delta \in \gamma\alpha\beta) \leftrightarrow (\mathbf{p}fd \Vdash \delta \in \alpha) \wedge (\mathbf{p}(\mathbf{p}_0g)(\iota_{\mathbf{t}}d(\mathbf{p}_1g)) \Vdash \delta \in \beta).$$

Thus it is completely analogous to verify that the biconditional is realised. $\square$

Therefore, we can conclude that:

Theorem 5.3.5. *Let φ be any theorem of CZF, then KP proves $\mathbf{V}_A \models \varphi$ for any subset $A \subseteq \omega$, i.e. there exists $d \in \mathbf{Kl}$ such that $d \Vdash \varphi$.*

Finally, the reason we need this new realisability interpretation is that the semantics we give $\mathbf{ML}_1\mathbf{V}$ here ensures every type is a subset of ω , so that the realisability model admits the following *axiom of subcountability*:

Lemma 5.3.6. *$\mathbf{V}_A$ realises that*

$$\forall x \exists u, f (u \subseteq \omega \wedge f \text{ is a surjection from } u \text{ onto } x).$$

Proof. For any $\alpha \in \mathbf{V}_A$ that is assigned to x , we can compute u as

$$\mathbf{p}(\mathbf{p}_0\alpha)(\lambda x. x^V),$$

and f as $\mathbf{p}(\mathbf{p}_0\alpha)(\lambda x. \langle x^V, \mathbf{p}_1\alpha x \rangle^V)$, where $\langle \beta, \gamma \rangle^V = \left\{ \{\beta, \beta\}^V, \{\beta, \gamma\}^V \right\}^V$. Observe that $\lambda x. \mathbf{p}x\iota_r \Vdash u \subseteq \omega^V$, and it is also easy to realise that f is a surjection from u onto x . $\square$

In the next section, we shall use the axiom of subcountability together with the distinguished primitive type $\overline{X}$ to construct an incomparable coding for all sets in this realisability model.

5.4 An equiconsistency proof for $\text{CZF} + V = L$

Following [49], we now select a specific set $A \subseteq \omega$ and see what effect it has on the realisability interpretation. To begin, we fix an arbitrary, recursive way to encode finite ω -tuples as a single natural number, such that the following notions are definable as applicative terms in the pca:

- $\mathbf{l}_s t = |t|$ is the length of the tuple,
- $\mathbf{t}_s t n$ is the truncated tuple of first n components in t for any $0 \leq n \leq \mathbf{l}_s t$,

whenever t is a valid code for some tuple. For example, we can encode the tuple $\langle a_0, \dots, a_{n-1} \rangle$ as

$$\mathbf{p}n(\mathbf{p}a_{n-1}(\mathbf{p}a_{n-2}(\dots \mathbf{p}a_1(\mathbf{p}a_0 0) \dots))),$$

conveniently reversed for back truncations, so that $\mathbf{l}_s = \mathbf{p}_0$, and

$$\mathbf{t}_s = \lambda t n. \mathbf{p}n(f(\mathbf{p}_0 t - n)(\mathbf{p}_1 t)),$$

where $f = \mathbf{r}(\lambda a g t. \mathbf{d}_{\mathbb{N}}^* a 0 t (g(\mathbf{p}_{\mathbb{N}} a)(\mathbf{p}_1 t)))$ strips a number of leading components from the given tuple.

We also need an alternative, recursive pairing function $\mathbf{p}'$, which has a specific property. We define it directly as follows:

$$\mathbf{p}'ab = \begin{cases} \max\{a, b\} \cdot (\max\{a, b\} + 1) - a + b & \text{if } \max\{a, b\} \text{ is even,} \\ \max\{a, b\} \cdot (\max\{a, b\} + 1) + a - b & \text{otherwise.} \end{cases}$$

We first observe that:

Lemma 5.4.1 (KP). *For any $n \in \omega$, the map $\langle a, b \rangle \mapsto \mathbf{p}'ab$ is a bijection between the Cartesian product $n \times n$ and $n^2 \in \omega$.*

Proof. Just observe that for any $a, b \in \omega$,

$$\max\{a, b\}^2 \leq \mathbf{p}'ab < (\max\{a, b\} + 1)^2,$$

so for each $n \in \omega$, $n > 0$, $\mathbf{p}'$ bijects precisely the set of $(2 \cdot n - 1)$ pairs

$$\{\langle n - 1, i \rangle : 0 \leq i < n\} \cup \{\langle i, n - 1 \rangle : 0 \leq i < n - 1\}$$

onto $\{(n - 1)^2, \dots, n^2 - 1\}$. □

It follows that $\langle a, b \rangle \mapsto \mathbf{p}'ab$ is a bijection $\omega^2 \rightarrow \omega$. Since $\mathbf{p}'$ is a total, recursive function, it is also a recursive process to search for the unique numbers $a, b \in \omega$ such that $\mathbf{p}'ab = x$ for a given $x \in \omega$. Therefore, by Lemma 2.2.1, it follows that we have the corresponding unpairing functions $\mathbf{p}'_0, \mathbf{p}'_1 \in \text{Kl}$ and we do not need to specify concrete mathematical expressions to compute them. Additionally, this specific $\mathbf{p}'$ is chosen to satisfy the following property:

Lemma 5.4.2. *For any $i, j \in \omega$ where $i \neq j$, there exist arbitrarily large $n \in \omega$ such that $\mathbf{p}'in < \mathbf{p}'jn$.*

Proof. It is easy to check that if $i > j$, then for any even $n > \max\{i, j\}$ we will have $\mathbf{p}'in < \mathbf{p}'jn$; otherwise, $i < j$ and for any odd $n > \max\{i, j\}$ we will have $\mathbf{p}'in < \mathbf{p}'jn$. □

We now build the infinite set A in the style of some recursion-theoretic priority argument, as the set of an increasing sequence of finite tuples

$$t_0 \sqsubseteq t_1 \sqsubseteq t_2 \sqsubseteq \dots$$

where each $t_i \in \omega^{<\omega}$ and $t_i \sqsubseteq t_j$, or equivalently, $\mathbf{l}_s t_i \leq \mathbf{l}_s t_j$ and $\mathbf{t}_s t_j(\mathbf{l}_s t_i) = t_i$ for any $i \leq j$. We then propose a countable number of requirements on finite tuples: for any $i, j, f \in \omega$ where $i \neq j$, we say that $t \in \omega^{<\omega}$ satisfies the requirement with index $\langle i, j, f \rangle$ if there exists some $n \in \omega$ such that $i, j < n$, $\mathbf{p}'in, \mathbf{p}'jn \leq \mathbf{l}_s t$, and

$$f(\mathbf{t}_s t(\mathbf{p}'in)) \downarrow \rightarrow f(\mathbf{t}_s t(\mathbf{p}'in)) \neq \mathbf{t}_s t(\mathbf{p}'jn),$$

i.e. that f , as a function in the pca Kl , does not correctly compute the first

$\mathbf{p}'jn$ components of the sequence t given its first $\mathbf{p}'in$ components, for some $n \in \omega$ where t has enough components. Clearly if some tuple t satisfies a requirement here, then every extension of t also satisfies it; we would like that for every such requirement, A contains a tail of tuples satisfying it. Formally, we prove the following statement:

Proposition 5.4.3 (KP). *There is some $h : \omega \rightarrow \omega$ such that for any $i, j, f \in \omega$ where $i \neq j$, the requirement with index $\langle i, j, f \rangle$ is satisfied by some finite initial segment of h .*

Proof. As we mentioned, the collection of finite tuples $t \in \omega^{<\omega}$ satisfying each single requirement is an upwards closed set, and Lemma 5.4.2 ensures that this set will be dense: for any $i, j \in \omega$ where $i \neq j$ and any tuple t , take some $n > i, j, \mathbf{l}_s t$ such that $\mathbf{p}'in < \mathbf{p}'jn$. It follows that

$$\mathbf{p}'jn > \mathbf{p}'in \geq n^2 > \mathbf{l}_s t$$

by Lemma 5.4.1. We extend t to an arbitrary $t' \sqsupseteq t$ such that $\mathbf{p}'in \leq \mathbf{l}_s t' < \mathbf{p}'jn$. Then for any further extension $t'' \sqsupseteq t'$, the value $f(\mathbf{t}_s t''(\mathbf{p}'in)) = f(\mathbf{t}_s t'(\mathbf{p}'in))$ (or the non-termination thereof) is already determined. Clearly by choosing a suitable extension t'' of length $\mathbf{p}'jn$, different from the output of f , we can make sure that f does not compute $\mathbf{t}_s t''(\mathbf{p}'jn) = t''$ correctly.

Therefore, we can now construct an increasing sequence of tuples

$$t_0 \sqsubseteq t_1 \sqsubseteq t_2 \sqsubseteq \dots$$

where t_0 is the empty tuple and each t_{i+1} is an extension of t_i found using the procedure above to satisfy the requirement with index $\langle \mathbf{p}_{00}i, \mathbf{p}_{01}i, \mathbf{p}_1i \rangle$ if $\mathbf{p}_{00}i \neq \mathbf{p}_{01}i$, and simply $t_{i+1} = t_i \frown 0$ if $\mathbf{p}_{00}i = \mathbf{p}_{01}i$. This will not involve choice principles since at each step, we can always take an appropriate t_{i+1} whose code is the least in ω . By Δ_0 -separation, clearly there exists $h : \omega \rightarrow \omega$ such that $h(n)$ is the n^{th} component in sufficiently long tuples in this sequence. $\square$

The distinguished type $\overline{X}$ shall then be given by the set

$$A = \{t \in \omega : t \sqsubseteq h\},$$

i.e. the (codes of) finite initial segments of h . We now examine

$$\alpha_0 = \mathbf{p}\overline{X}\left(\lambda t. (\mathbf{l}_s t)^V\right) \in \mathbf{V}_A$$

in the realisability model, and see that it has the following basic properties:

Lemma 5.4.4. $\mathbf{V}_A$ realises the following:

- (i) $\alpha_0 \in \text{Ord}$;
- (ii) $\alpha_0 \subseteq \omega$.

Proof. First check that

$$\lambda t. \mathbf{p}(\mathbf{l}_s t)_{\iota_r} \Vdash \forall \alpha \in \alpha_0 \alpha \in \omega^V,$$

thus (ii) is realised.

Now, for (i), observe that for any $t \in A$ and $0 \leq n \leq \mathbf{l}_s t$, $\mathbf{p}(\mathbf{t}_s t n)_{\iota_r} \Vdash n^V \in \alpha_0$. Therefore,

$$\lambda t x. \mathbf{p}(\mathbf{t}_s t x)_{\iota_r} \Vdash \forall \alpha \in \alpha_0 \forall \beta \in \alpha \beta \in \alpha_0,$$

realising that α_0 is transitive. Now it follows immediately from $\alpha_0 \subseteq \omega$ within CZF that any element in α_0 is transitive, thus $\alpha_0 \in \text{Ord}$. $\square$

Next, we use the fact that our pairing function $\mathbf{p}'$ is recursive; this means we can write down the same definition inside the realisability model (or even inside L) to define a set-theoretic function from ω^2 to ω that performs exactly the same computations. More explicitly, we construct the following name of a set-theoretic pairing function

$$\bar{p} = \mathbf{p}\bar{\omega} \left(\lambda x. \left\langle \left\langle (\mathbf{p}'_0 x)^V, (\mathbf{p}'_1 x)^V \right\rangle^V, x^V \right\rangle^V \right) \in \mathbf{V}_A,$$

then the realisability model will interpret $\bar{p}(i^V, j^V) = (\mathbf{p}'ij)^V$ for the names $i^V, j^V, (\mathbf{p}'ij)^V \in \mathbf{V}_A$, for any $i, j \in \omega$.

We use $\bar{p}$ inside the realisability model to extract the complexity coded into α_0 in Proposition 5.4.3, by defining a set-theoretic function $f_0 : \omega \rightarrow \mathcal{P}(\omega)$ as

$$f_0(i) = \{n \in \omega : \forall k \in n \bar{p}(i, k) \in \alpha_0\}$$

within $\mathbf{V}_A$. It is easy to check that this definition guarantees that $\forall i \in \omega f_0(i) \in \mathcal{P}(\omega) \cap \text{Ord}$ is realised. We want to invoke Corollary 5.1.10 on f_0 , so we check that it is realised to be pairwise strongly incomparable:

Proposition 5.4.5. *For any $i, j \in \omega$, if some $d \Vdash f_0(i^V) \subseteq f_0(j^V)$, then*

$i = j$. Namely, it follows immediately that

$$\lambda i j x. \iota_r \Vdash \forall i, j \in \omega (f_0(i) \subseteq f_0(j) \rightarrow i = j).$$

Proof. Assume otherwise, i.e. $i \neq j$. Reasoning inside $\mathbf{V}_A$, we know that $f_0(i^V) \subseteq f_0(j^V)$, by definition of f_0 , is equivalent to

$$\forall n \in \omega^V (\forall k \in n \bar{p}(i^V, k) \in \alpha_0 \rightarrow \forall k \in n \bar{p}(j^V, k) \in \alpha_0),$$

so suppose that this is realised by some $d \in \text{Kl}$. Pedantically, we take the formula $\bar{p}(i^V, k^V) \in \alpha_0$ as a shorthand for

$$\exists x \in \bar{p} \exists n \in \alpha_0 x = \langle \langle i^V, k^V \rangle, n \rangle,$$

thus if some $e \Vdash \bar{p}(i^V, k^V) \in \alpha_0$, then $\mathbf{p}_0 e = \mathbf{p}'ik$, and $\mathbf{p}_{10}e$ must be the unique finite sequence in A of length precisely $\mathbf{p}'ik$.

Conversely, we can also find some function $q \in \text{Kl}$ such that, suppose $t \in A$ is any finite sequence satisfying $\mathbf{l}_s t \geq \mathbf{p}'ik$, then

$$\mathbf{p}(\mathbf{p}'ik)(\mathbf{p}(\mathbf{t}_s t(\mathbf{p}'ik))(qik)) \Vdash \bar{p}(i^V, k^V) \in \alpha_0.$$

(Here we use q to conceal the cumbersome process of writing out the internal structure of realisers for the defining formula of set-theoretic ordered pairs, but it is routine to check that this can be done.)

Consequently, for any $n > \max\{i, j\}$, we let $t \in A$ be any long enough finite sequence such that $\mathbf{p}'in, \mathbf{p}'jn \leq \mathbf{l}_s t$. Observe that our stipulation of the pairing function $\mathbf{p}'$ also ensures $\forall k \leq n \mathbf{p}'ik \leq \mathbf{p}'in$. Therefore,

$$\lambda k. \mathbf{p}(\mathbf{p}'ik)(\mathbf{p}(\mathbf{t}_s t(\mathbf{p}'ik))(qik)) \Vdash \forall k \in (\mathbf{s}_{\mathbb{N}} n)^V \bar{p}(i^V, k) \in \alpha_0.$$

In other words, if we pass this to the realiser d at the beginning of this proof, i.e. we take $f = \lambda n t. d(\mathbf{s}_{\mathbb{N}} n)(\lambda k. \mathbf{p}(\mathbf{p}'ik)(\mathbf{p}(\mathbf{t}_s t(\mathbf{p}'ik))(qik)))n$, then we have $fnt \Vdash \bar{p}(j^V, n^V) \in \alpha_0$ and $\mathbf{p}_{10}(fnt) = \mathbf{t}_s t(\mathbf{p}'jn)$.

Observe that $\mathbf{l}_s(\mathbf{t}_s t(\mathbf{p}'in)) = \mathbf{p}'in$, i.e. $\mathbf{p}'_1(\mathbf{l}_s(\mathbf{t}_s t(\mathbf{p}'in))) = n$. This means we can have a single-parameter function $g = \lambda s. \mathbf{p}_{10}(f(\mathbf{p}'_1(\mathbf{l}_s s))s)$, such that

$$g(\mathbf{t}_s t(\mathbf{p}'in)) \downarrow \wedge g(\mathbf{t}_s t(\mathbf{p}'in)) = \mathbf{t}_s t(\mathbf{p}'jn).$$

However, this recursive g is now universal for any appropriate n and long

enough $t \in A$, whereas in our construction of A through Proposition 5.4.3 we have required that there exist n and $t \in A$ where g does not compute $\mathbf{t}_s t(\mathbf{p}'jn)$ from $\mathbf{t}_s t(\mathbf{p}'in)$ correctly. Therefore, $i \neq j$ implies a contradiction and thus we must have $i = j$. $\square$

Finally, it follows from the remark before Corollary 5.1.10 that α_0 , as an element of $\mathcal{P}(\omega) \cap \text{Ord}$, satisfies

$$\mathbf{V}_A \models \alpha_0 = L_{\alpha_0} \cap \text{Ord} \in L.$$

Therefore, f_0 , which is defined using parameters ω (which lies in L by Corollary 4.3.5), $\bar{p}$ (which is recursively definable from ω) and $\alpha_0 \in L$, also lies in L . By Corollary 5.1.10, we have $\mathbf{V}_A \models \mathcal{P}(\omega) \subseteq L$.

Theorem 5.4.6. $\mathbf{V}_A$ realises $V = L$.

Proof. We work inside $\mathbf{V}_A$, which is a model of CZF + subcountability by section 5.3.

For any set s consider the transitive closure $\text{TC}(\{s\})$ and correspondingly a surjection $f : u \rightarrow \text{TC}(\{s\})$ where $u \subseteq \omega$, by the axiom of subcountability. Without loss of generality, we can assume conveniently that $0 \in u$ and $f(0) = s$. We define

$$\sigma_s = \{n \in \omega : \exists i, j \in u (f(i) \in f(j) \wedge n = \bar{p}(i, j))\}.$$

Given σ_s , we can recursively define some (possibly partial) function g on domain u as

$$g(n) = \{g(m) : m \in u \wedge \bar{p}(m, n) \in \sigma_s\}$$

if $g(m)$ exists for all such $m \in u$. The function g is defined by a Σ_1 -formula using parameters u, σ_s , where $u, \sigma_s \in L$ because $\mathbf{V}_A$ realises $\mathcal{P}(\omega) \subseteq L$. Using collection, we can easily prove by set induction on x that, for any set x ,

$$\forall n \in u (f(n) = x \rightarrow g(n) = x \wedge L \models g(n) \text{ exists}).$$

Since g , as a Σ_1 -definable function, must be upwards absolute between transitive classes, clearly $s = f(0) = g(0) \in L$. $\square$

Corollary 5.4.7. *The consistency of KP implies that of CZF + $V = L$. It*

follows that

$$\text{CZF} + V = L \equiv_{\text{Con}} \text{KP} \equiv_{\text{Con}} \text{CZF}.$$

Proof. This follows immediately from Theorem 5.3.5 and Theorem 5.4.6. $\square$

Chapter 6

Plump ordinals and powersets

From chapter 5, we observe that the primary difficulty when using the method of incomparable codings to work with L is that, given a collection of incomparable ordinals $s \subseteq L \cap \text{Ord}$ and an arbitrary subset $u \subseteq s$, it can be hard to ensure the single ordinal that can be used to uniquely identify this set, $\bigcup_{\alpha \in u} \alpha^+$, is guaranteed to lie in L without knowing that $u \in L$, essentially due to Problem 4.3.3 that there is no known proof of $\text{Ord} \subseteq L$ in intuitionistic logic. Consequently, we often need to resort to more complicated, non-systematic constructions such as previously done in Proposition 5.1.9 and Corollary 5.1.10.

In this chapter, we will formalise the notion of *plump ordinals*, originally discussed in a category-theoretic context by Paul Taylor [45]. We show that assuming the powerset axiom, the unbounded subclass of plump ordinals will form a smaller ‘class of ordinals’ that are always in L , and we will see how this simplifies arguments of incomparable codings. Using a weak realisability interpretation of $\text{CZF}(\mathcal{P})$ inside classical $\text{KP}(\mathcal{P})$, we will then extend the equiconsistency proof in section 5.4 to $\text{CZF}(\mathcal{P}) + V = L$.

6.1 Theory of plump ordinals

To begin, we present the general results on plump ordinals from my manuscript [50]. Following Taylor [45], we informally say that a class $\mathcal{O} \subseteq \text{Ord}$ is *the class of plump ordinals* if it contains precisely those ordinals $\alpha \in \text{Ord}$ satisfying the following two conditions:

- (1) $\alpha \subseteq \mathcal{O}$,
- (2) for any $\beta \in \alpha$, the class $\mathcal{P}(\beta) \cap \mathcal{O} \subseteq \alpha$.

Formally speaking, this is not a definition for the class $\mathcal{O}$, because it is self-referential and some form of recursion theorem is needed to justify that such a class $\mathcal{O}$ is indeed definable. But first observe that the class of plump ordinals, if it exists, is at least unique:

Proposition 6.1.1. *Suppose that $\mathcal{O}_1, \mathcal{O}_2$ are both ‘classes of plump ordinals’ as just defined, then $\mathcal{O}_1 = \mathcal{O}_2$.*

Proof. It suffices to prove by set induction that for any $\alpha \in \mathcal{O}_1 \cup \mathcal{O}_2$, the classes

$$\mathcal{P}(\alpha) \cap \mathcal{O}_1 = \mathcal{P}(\alpha) \cap \mathcal{O}_2.$$

To begin, assume that $\alpha \in \mathcal{O}_1$, and we shall first show using the inductive hypothesis that $\mathcal{P}(\alpha) \cap \mathcal{O}_1 \subseteq \mathcal{P}(\alpha) \cap \mathcal{O}_2$. Consider any $\beta \subseteq \alpha$ such that $\beta \in \mathcal{O}_1$, then for any $\gamma \in \beta \subseteq \alpha$, we also have $\gamma \in \mathcal{O}_1$. By the inductive hypothesis,

$$\gamma \in \mathcal{P}(\gamma) \cap \mathcal{O}_1 = \mathcal{P}(\gamma) \cap \mathcal{O}_2,$$

i.e. $\gamma \in \mathcal{O}_2$ as well. Therefore, $\beta \subseteq \mathcal{O}_2$.

Additionally, for any $\gamma \in \beta \subseteq \alpha$, we have $\mathcal{P}(\gamma) \cap \mathcal{O}_1 \subseteq \beta$. Thus again by the inductive hypothesis,

$$\mathcal{P}(\gamma) \cap \mathcal{O}_2 = \mathcal{P}(\gamma) \cap \mathcal{O}_1 \subseteq \beta.$$

Therefore, both criteria are verified, and by the definition of $\mathcal{O}_2$, we must also have $\beta \in \mathcal{O}_2$.

Now, by an entirely symmetrical argument, we can show that $\alpha \in \mathcal{O}_2$ implies $\mathcal{P}(\alpha) \cap \mathcal{O}_2 \subseteq \mathcal{P}(\alpha) \cap \mathcal{O}_1$. Notice that as $\alpha \in \mathcal{P}(\alpha)$, this suffices for the conclusion that for any $\alpha \in \mathcal{O}_1 \cup \mathcal{O}_2$, we have

$$\alpha \in \mathcal{O}_1 \leftrightarrow \alpha \in \mathcal{O}_2.$$

In other words, $\alpha \in \mathcal{O}_1 \cup \mathcal{O}_2$ implies both $\alpha \in \mathcal{O}_1$ and $\alpha \in \mathcal{O}_2$, and thus $\mathcal{P}(\alpha) \cap \mathcal{O}_1 = \mathcal{P}(\alpha) \cap \mathcal{O}_2$ as desired. $\square$

Inside the theory $\text{IKP}(\mathcal{P})$ where $\mathcal{P}$ is a primitive function symbol that is allowed to occur in Σ_1 -formulae, it is not hard to imitate [45, Proposition 4.3] and use Σ -recursion on $\alpha \in \text{Ord}$ to define the family of sets $\theta_\alpha \subseteq \mathcal{P}(\alpha)$ satisfying

$$\theta_\alpha = \{\beta \in \mathcal{P}(\alpha) \cap \text{Ord} : \forall \gamma \in \beta (\gamma \in \theta_\gamma \wedge \theta_\gamma \subseteq \beta)\}$$

for each $\alpha \in \text{Ord}$. Then it is easy to verify that $\mathcal{O} = \{\alpha \in \text{Ord} : \alpha \in \theta_\alpha\}$ will be the class of plump ordinals.

However, for our purpose of studying the interaction of the plump ordinals and Gödel's constructible universe L , it can be difficult to adapt this construction since the powerset axiom is known to fail in L in extensions of $\text{IKP}(\mathcal{P})$ (cf. [25, section 7], and previous discussions in section 5.2). Thus, we will use the alternative presentation in [50] where we show that the class of plump ordinals can be defined by a Π_1 -formula — in fact, a Δ_0 -formula if we consider $\mathcal{P}$ as a primitive function symbol — in any extension of IKP .

To this end, we denote

$$\text{relpl}_\alpha(\gamma) \quad :\Longleftrightarrow \quad \forall \delta \in \gamma \forall \varepsilon \in \alpha (\varepsilon \subseteq \delta \rightarrow \varepsilon \in \gamma)$$

and say that γ is *plump relative to* α when this holds. We define PlOrd as the Π_1 -subclass of $\alpha \in \text{Ord}$ satisfying

$$\forall \beta \in \alpha \forall \gamma \subseteq \beta (\text{relpl}_\alpha(\gamma) \rightarrow \gamma \in \alpha \wedge \forall \delta \in \alpha (\beta \in \delta \rightarrow \gamma \in \delta)).$$

We shall show that PlOrd is precisely the class of plump ordinals.

Let us first observe the following basic properties:

Lemma 6.1.2. *For any $\alpha \in \text{PlOrd}$, if $\beta \in \alpha$, then $\text{relpl}_\alpha(\beta)$ holds.*

Proof. We show by set induction on β that if $\beta \in \alpha$, then

$$\forall \gamma \in \alpha (\gamma \subseteq \beta \rightarrow \text{relpl}_\alpha(\gamma)).$$

Fix $\gamma, \beta \in \alpha$ such that $\gamma \subseteq \beta$, and assume that $\delta \in \gamma$, $\varepsilon \in \alpha$ and $\varepsilon \subseteq \delta$. It follows that $\delta \in \beta$ (and $\delta \in \alpha$ by transitivity), thus by the inductive hypothesis $\text{relpl}_\alpha(\varepsilon)$ holds. We know that $\varepsilon \subseteq \delta$, $\gamma \in \alpha$ and $\delta \in \gamma$, then the assumption $\alpha \in \text{PlOrd}$ must imply that $\varepsilon \in \gamma$, precisely what we need to conclude that $\text{relpl}_\alpha(\gamma)$ holds. $\square$

Lemma 6.1.3. *For any $\alpha \in \text{PlOrd}$, if $\beta \subseteq \alpha$ and $\text{relpl}_\alpha(\beta)$ holds, then $\beta \in \text{PlOrd}$.*

Proof. First notice that $\beta \subseteq \alpha$ and $\text{relpl}_\alpha(\beta)$ imply that β is transitive: consider any $\gamma \in \beta$, $\delta \in \gamma$, we know that $\delta \in \alpha$ since $\alpha \in \text{Ord}$ and also $\delta \subseteq \gamma$ since $\gamma \in \text{Ord}$. It follows from $\text{relpl}_\alpha(\beta)$ that we must have $\delta \in \beta$.

Now suppose that $\beta' \in \beta$, $\gamma \subseteq \beta'$ and $\text{relpl}_\beta(\gamma)$ holds. For any $\delta \in \gamma$ and $\varepsilon \in \alpha$ such that $\varepsilon \subseteq \delta$, we know by transitivity that $\delta \in \beta$, so it follows from $\text{relpl}_\alpha(\beta)$ that $\varepsilon \in \beta$, and it follows again from $\text{relpl}_\beta(\gamma)$ that $\varepsilon \in \gamma$. In other words, $\text{relpl}_\alpha(\gamma)$ holds.

From the assumption that $\alpha \in \text{PIOrd}$, we then know that $\gamma \in \alpha$. Since also $\beta' \in \beta$ and $\gamma \subseteq \beta'$, it follows from $\text{relpl}_\alpha(\beta)$ that $\gamma \in \beta$. For any $\delta \in \beta$, we now have $\beta' \in \alpha$, $\gamma \subseteq \beta'$ and $\delta \in \alpha$, so it follows from $\text{relpl}_\alpha(\gamma)$ that $\beta' \in \delta$ implies $\gamma \in \delta$. We have shown that $\beta \in \text{PIOrd}$. $\square$

Proposition 6.1.4. *For any $\alpha \in \text{PIOrd}$, if $\beta \in \alpha$, then $\beta \in \text{PIOrd}$.*

Proof. We have $\text{relpl}_\alpha(\beta)$ by Lemma 6.1.2, and we also have $\beta \subseteq \alpha$ by transitivity. Thus we have $\beta \in \text{PIOrd}$ by Lemma 6.1.3. $\square$

Lemma 6.1.5. *For any $\alpha, \beta \in \text{PIOrd}$, if $\beta \subseteq \alpha$, then $\text{relpl}_\alpha(\beta)$ holds.*

Proof. Suppose that $\delta \in \beta$, $\varepsilon \in \alpha$ and $\varepsilon \subseteq \delta$. By Lemma 6.1.2, we know that $\text{relpl}_\alpha(\varepsilon)$ holds. Since $\beta \subseteq \alpha$, it is easy to check that $\text{relpl}_\beta(\varepsilon)$ also holds. Then the assumption that $\beta \in \text{PIOrd}$ immediately implies $\varepsilon \in \beta$, precisely what we need to show. $\square$

Proposition 6.1.6. *For any $\alpha, \gamma \in \text{PIOrd}$, if $\gamma \subseteq \beta$ for some $\beta \in \alpha$, then $\gamma \in \alpha$.*

Proof. Since $\alpha \in \text{PIOrd}$, it suffices to show that $\text{relpl}_\alpha(\gamma)$ holds. But we know that $\gamma \subseteq \beta \subseteq \alpha$ by transitivity, thus Lemma 6.1.5 suffices for the proof. $\square$

Here Proposition 6.1.4 states that any $\alpha \in \text{PIOrd}$ satisfies the aforementioned condition (1) for the ‘class of plump ordinals’, whereas Proposition 6.1.6 states that any $\alpha \in \text{PIOrd}$ satisfies condition (2). For the converse, we need to verify that:

Proposition 6.1.7. *For any set x , we have $x \in \text{PIOrd}$ if both of the following hold:*

- (1) *for any $\beta \in x$, $\beta \in \text{PIOrd}$;*
- (2) *for any $\gamma \in \text{PIOrd}$, if $\gamma \subseteq \beta$ for some $\beta \in x$, then $\gamma \in x$.*

Proof. Condition (1) already ensures that $x \subseteq \text{Ord}$. For transitivity, note that for any $\beta \in x$, $\gamma \in \beta$, we know that $\gamma \subseteq \beta$ since $\beta \in \text{Ord}$, and $\gamma \in \text{PIOrd}$ by Proposition 6.1.4, thus condition (2) implies that $\gamma \in x$ as well.

Now suppose that $\beta \in x$, $\gamma \subseteq \beta$ and $\text{relpl}_x(\gamma)$ holds. Since $\beta \subseteq x$ by transitivity, it follows immediately from definition that $\text{relpl}_\beta(\gamma)$ also holds. Here $\beta \in \text{PlOrd}$, so by Lemma 6.1.3, $\gamma \in \text{PlOrd}$, thus condition (2) ensures that $\gamma \in x$.

Additionally, for any $\delta \in x$, suppose that $\beta \in \delta$. Observe that we also have $\text{relpl}_\delta(\gamma)$ and $\delta \in \text{PlOrd}$, so it follows immediately that $\gamma \in \delta$ as well. $\square$

This allows us to finally conclude that:

Corollary 6.1.8 (IKP). *PlOrd is the class of plump ordinals.*

Observe that combining Lemma 6.1.3 and Lemma 6.1.5, we know that for any $\alpha \in \text{PlOrd}$ and any $\beta \subseteq \alpha$,

$$\beta \in \text{PlOrd} \leftrightarrow \text{relpl}_\alpha(\beta),$$

where the right-hand side of the equivalence is a Δ_0 -formula. Therefore, for any $\alpha \in \text{PlOrd}$, we can define its *plump successor* as

$$\alpha^{\text{pl}+} = \{\beta \in \mathcal{P}(\alpha) : \text{relpl}_\alpha(\beta)\} = \mathcal{P}(\alpha) \cap \text{PlOrd}$$

if this exists as a set; and we know in IKP that whenever the powerset $\mathcal{P}(\alpha)$ exists, the plump successor $\alpha^{\text{pl}+}$ also exists by Δ_0 -separation. We verify that:

Lemma 6.1.9. *For any $\alpha \in \text{PlOrd}$, if $\alpha^{\text{pl}+}$ exists, then $\alpha^{\text{pl}+} \in \text{PlOrd}$.*

Proof. We just need to check the two criteria in Proposition 6.1.7: for any $\beta \in \alpha^{\text{pl}+}$, we have $\beta \in \text{PlOrd}$ by definition; for any $\gamma \in \text{PlOrd}$, if $\gamma \subseteq \beta$ for some $\beta \in \alpha^{\text{pl}+}$, then $\gamma \subseteq \alpha$ and $\gamma \in \alpha^{\text{pl}+}$ again by definition. $\square$

Lemma 6.1.10. *For any set $s \subseteq \text{PlOrd}$, $\bigcup s \in \text{PlOrd}$.*

Proof. We again check the two criteria in Proposition 6.1.7: for any $\beta \in \bigcup s$, we have $\beta \in \alpha$ for some $\alpha \in s \subseteq \text{PlOrd}$. Then $\beta \in \text{PlOrd}$ by Proposition 6.1.4. For any $\gamma \in \text{PlOrd}$, if $\gamma \subseteq \beta$ for some $\beta \in \bigcup s$, then again we have $\beta \in \alpha$ for some $\alpha \in s \subseteq \text{PlOrd}$, and it follows that $\gamma \in \alpha \subseteq \bigcup s$ by Proposition 6.1.6. $\square$

Note that IKP alone, however, does not suffice to prove that many plump ordinals exist. We can observe that:

Lemma 6.1.11. *Any set $\alpha \subseteq 1 = \{\emptyset\}$ is a plump ordinal.*

Proof. Check the two criteria in Proposition 6.1.7: for any $\beta \in \alpha$, we have $\beta = \emptyset$ and clearly $\beta \in \text{PlOrd}$ by definition; for any $\gamma \subseteq \beta$, we also have $\gamma = \beta = \emptyset$ and thus $\gamma \in \alpha$ as well. $\square$

It follows that $1 \in \text{PlOrd}$ and also $1^{\text{pl}+} = \mathcal{P}(1) \in \text{PlOrd}$ (if it exists) by Lemma 6.1.9. So IKP cannot even prove that $1^{\text{pl}+}$ exists.

In contrast, in $\text{IKP}(\mathcal{P})$ or any extension of it, we can define class functions by Σ -recursion on a defining formula that contains the primitive symbol $\mathcal{P}$, so we can define the *plump operator* $(-)^{\text{pl}} : \text{Ord} \rightarrow \text{PlOrd}$ recursively as

$$\alpha^{\text{pl}} = \bigcup_{\beta \in \alpha} (\beta^{\text{pl}})^{\text{pl}+}.$$

It follows from Lemma 6.1.9 and Lemma 6.1.10 that this is well-defined and $\alpha^{\text{pl}} \in \text{PlOrd}$ for every $\alpha \in \text{Ord}$. It is easy to check that:

Proposition 6.1.12. *For any $\alpha \in \text{PlOrd}$, $\alpha^{\text{pl}} = \alpha$. Thus $(-)^{\text{pl}}$ is a surjection onto PlOrd .*

Proof. It follows immediately from Proposition 6.1.6 that for any $\alpha, \beta \in \text{PlOrd}$, if $\beta \in \alpha$ then also $\beta^{\text{pl}+} \subseteq \alpha$. Thus, by the inductive hypothesis, we have

$$\alpha^{\text{pl}} = \bigcup_{\beta \in \alpha} (\beta^{\text{pl}})^{\text{pl}+} = \bigcup_{\beta \in \alpha} \beta^{\text{pl}+} = \alpha$$

for any $\alpha \in \text{PlOrd}$. $\square$

Plump ordinals and L

We now proceed to demonstrate that, unlike the complicated scenario with arbitrary ordinals and L in intuitionistic logic, as we presented in section 4.4, the behaviours of plump ordinals are much simpler to characterise. We will always have that:

Lemma 6.1.13 (IKP, [50, Lemma 3.5]). *For any $\alpha \in \text{PlOrd}$, we have*

$$\alpha = \{x \in L_\alpha : L_\alpha \models x \in \text{PlOrd}\} \in \text{def}(L_\alpha).$$

Proof. We shall prove this by set induction on α . Fix some $\alpha \in \text{PlOrd}$, then the inductive hypothesis says that for any $\beta \in \alpha$,

$$\beta = \{x \in L_\beta : L_\beta \models x \in \text{PlOrd}\} \in \text{def}(L_\beta),$$

so immediately $\alpha \subseteq L_\alpha$. Notice additionally that PIOrd is a Π_1 -class, which is downwards absolute between transitive sets and classes, thus for any $\beta \in \alpha$, since $\beta \in \text{PIOrd}$, we must also have $L_\alpha \models \beta \in \text{PIOrd}$.

It remains to prove the converse: for any $x \in L_\alpha$ such that $L_\alpha \models x \in \text{PIOrd}$, we want to show that $x \in \alpha$. There must exist some $\beta \in \alpha$ such that $x \in \text{def}(L_\beta)$. Consider any $y \in x$, we observe that Lemma 6.1.2 and Lemma 6.1.3 use no other assumptions than the axiom of set induction, thus their relativisations to any transitive set still hold, and we can relativise Proposition 6.1.4:

$$L_\alpha \models \forall x \in \text{PIOrd} \forall y \in x \ y \in \text{PIOrd}.$$

Here, α is transitive, thus $\beta \subseteq \alpha$ and $L_\beta \subseteq L_\alpha$. Again by downward absoluteness, we know that $y \in L_\beta$ satisfies $L_\beta \models y \in \text{PIOrd}$. By the inductive hypothesis, this means $y \in \beta$, i.e. $x \subseteq \beta$.

Finally, we observe that Lemma 6.1.5 also relativises to the transitive set L_α , thus we have $L_\alpha \models \text{relpl}_\beta(x)$. However, $\text{relpl}_\beta(x)$ is a Δ_0 -formula, so it also holds in the original universe V . Thus, $x \in \text{PIOrd}$ in V by Lemma 6.1.3, and since $x \subseteq \beta$, $\alpha \in \text{PIOrd}$ immediately implies that $x \in \alpha$. $\square$

Corollary 6.1.14 (IKP). $\text{PIOrd} = \text{PIOrd}^L$.

Proof. For any $\alpha \in \text{PIOrd}$, by Lemma 6.1.13, $\alpha \in \text{def}(L_\alpha)$, thus $\alpha \in L_{\alpha^+} \in L$. We also have $L \models \alpha \in \text{PIOrd}$ by downward absoluteness, thus $\text{PIOrd} \subseteq \text{PIOrd}^L$.

Conversely, we just show that $\alpha \in \text{PIOrd}^L$ implies $\alpha \in \text{PIOrd}$ by set induction on α . Using the fact that $L \models \text{IKP}$ (cf. section 5.2), we know that our lemmata on plump ordinals all work inside L . Thus, assuming $L \models \alpha \in \text{PIOrd}$, it follows that for any $\beta \in \alpha$, we have $L \models \beta \in \text{PIOrd}$ by Proposition 6.1.4 and thus actually $\beta \in \text{PIOrd}$ by the inductive hypothesis.

Additionally, for any $\beta \in \alpha$ and $\gamma \in \text{PIOrd}$ such that $\gamma \subseteq \beta$, we have $\gamma \in L$ by the forward direction and thus $\gamma \in \alpha$ since $L \models \alpha \in \text{PIOrd}$. It follows from Proposition 6.1.7 that $\alpha \in \text{PIOrd}$ indeed. $\square$

In [50], we further considered the class⁴¹

$$L_{\text{pl}} = \bigcup_{\alpha \in \text{PIOrd}} L_\alpha \subseteq L$$

⁴¹It remains open, however, whether $L \neq L_{\text{pl}}$ is a consistent axiom in any intuitionistic set theory.

and verified that also $\text{PlOrd} = \text{PlOrd}^{L_{\text{pl}}}$. It is argued that, when T denotes IKP together with a technical axiom scheme of *plump* Σ_1 -*bounding*, $\text{PlB}\Sigma_1$,

$$\forall x \in a \exists \alpha \in \text{PlOrd} \varphi(x, \alpha) \rightarrow \exists \alpha \in \text{PlOrd} \forall x \in a \exists \beta \in \alpha \varphi(x, \beta)$$

for any Σ_1 -formula φ , then L_{pl} can be seen as some kind of ‘inner model’ for T , in the sense that T verifies both of the following:

- (1) L_{pl} satisfies $T = \text{IKP} + \text{PlB}\Sigma_1$ and also $V = L_{\text{pl}}$;
- (2) L_{pl} contains all plump ordinals.

It is worth noting that restricted excluded middle, REM, implies that ordinals are linearly ordered, and thus every ordinal is plump. Therefore $\text{IKP} + \text{REM}$ implies that $L_{\text{pl}} = L$ and thus, by an entirely analogous argument as Proposition 5.2.3, we know that $\text{CZF}(\mathcal{P}) + \neg \text{Exp}^{L_{\text{pl}}}$ is consistent. It follows that if T is any fragment of $\text{CZF}(\mathcal{P})$ that proves Exp, then L_{pl} may no longer be an inner model in the sense above.

However, we will not go further into the details as the only key fact relevant to this thesis is Lemma 6.1.13, since it implies that every plump ordinal is automatically in L (or even L_{pl}), and thus if we work with an incomparable encoding where all coding objects are plump ordinals, then every coded set will be definable within L as well.

Plump ordinal arithmetic

It is apparent that the automatic process with the weak incomparables in section 5.1 cannot be replicated within the plump ordinals — especially there is no analogue of Lemma 5.1.2 where every involved ordinal is plump. However, there are still some semi-automatic processes that can simplify the construction and treatment of incomparable plump ordinals.

To present these results, we need to define a new notion of ordinal arithmetic that only produces plump ordinals, by replacing the successor operation used in usual ordinal arithmetic with the plump successor $(-)^{\text{pl}}$. Specifically, we denote

$$\begin{aligned} \alpha +^{\text{pl}} \beta &= \alpha \cup \bigcup_{\gamma \in \beta} (\alpha +^{\text{pl}} \gamma)^{\text{pl}+}, \\ \alpha \cdot^{\text{pl}} \beta &= \bigcup_{\gamma \in \beta} (\alpha \cdot^{\text{pl}} \gamma +^{\text{pl}} \alpha) \end{aligned}$$

(cf. the usual ordinal addition in intuitionistic logic presented in section 5.1). These operations are definable in $\text{IKP}(\mathcal{P})$ by Σ -recursion on formulae containing the primitive function symbol $\mathcal{P}$, or by recursion using unrestricted strong collection in CZF. Do notice that $\beta \in \text{Ord}$ suffices for the following:

Lemma 6.1.15. *For any $\alpha \in \text{PlOrd}$, $\beta \in \text{Ord}$, we have $\alpha +^{\text{pl}} \beta, \alpha \cdot^{\text{pl}} \beta \in \text{PlOrd}$.*

Proof. This follows from easy set induction on β , using Lemma 6.1.9 and Lemma 6.1.10. $\square$

Before we proceed, I want to make a technical remark on the recursive process involved here: since its defining formula invokes plumpness and thus the powerset operation, such definitions cannot be replicated as given inside L in $\text{IKP}(\mathcal{P})$ or CZF, because in L , Exp and thus the powerset axiom may fail. However, we should observe that, given fixed α and β , there will be a large enough plump ordinal σ which contains all the plump ordinals used in the process to define $\gamma +^{\text{pl}} \delta$ for all $\gamma \in \alpha$, $\delta \in \beta$ — namely, $\sigma = \alpha +^{\text{pl}} \beta$ will work. It then follows that in the defining formula of this recursive process, every occurrence of $\gamma \in \text{PlOrd}$ can be replaced by $\gamma \in \sigma$ and likewise, for example, every occurrence of $\gamma = \delta^{\text{pl}+}$ can be replaced by

$$\gamma \in \sigma \wedge \delta \in \gamma \wedge \forall \eta \in \gamma \eta \subseteq \delta.$$

It follows that plump ordinal addition up to α and β can be defined by a Σ_1 -formula

$$\gamma +^{\text{pl}} \delta = \eta \quad :\Longleftrightarrow \quad \varphi(\gamma, \delta, \eta; \sigma),$$

not containing the powerset symbol at all, but parametrised on σ . Since the plump ordinal parameter σ must lie in L , by Σ -recursion in IKP we know that this operation is well-defined with its existential witnesses all inside L . By Corollary 6.1.14, we can then formulate plump ordinal addition in general as

$$\alpha +^{\text{pl}} \beta = \eta \quad :\Longleftrightarrow \quad \exists \sigma \in \text{PlOrd} \varphi(\alpha, \beta, \eta; \sigma),$$

which will define the same relation when interpreted in both V and L . Similar constructions work for plump ordinal multiplication and also for the recursive procedure we later employ in Theorem 6.1.19, and we see that the same process can also be carried out in the subclass $L_{\text{pl}} = \bigcup_{\alpha \in \text{PlOrd}} L_\alpha$ as well.

Now, we check that the following basic properties of arithmetic hold. Observe that many short proofs here work specifically for plump ordinals and are not

valid for the usual ordinal arithmetic on Ord in an intuitionistic context:

Lemma 6.1.16. *For any $\alpha, \beta, \gamma \in \text{PlOrd}$, $\alpha +^{\text{pl}} \beta \subseteq \alpha +^{\text{pl}} \gamma$ implies $\beta \subseteq \gamma$. In other words, the class function $F_\alpha : \text{PlOrd} \rightarrow \text{PlOrd}$ given by $\beta \mapsto \alpha +^{\text{pl}} \beta$ is injective.*

Proof. We show this by set induction on γ . Consider any $\delta \in \beta$, then $\alpha +^{\text{pl}} \delta \in (\alpha +^{\text{pl}} \delta)^{\text{pl}+} \subseteq \alpha +^{\text{pl}} \beta$. By definition, $\alpha \subseteq \alpha +^{\text{pl}} \delta$, so $\alpha +^{\text{pl}} \delta \in \alpha$ would imply $\alpha \in \alpha$ by plumpness, thus a contradiction. Therefore, we must have $\alpha +^{\text{pl}} \delta \in (\alpha +^{\text{pl}} \varepsilon)^{\text{pl}+}$ for some $\varepsilon \in \gamma$. This means $\alpha +^{\text{pl}} \delta \subseteq \alpha +^{\text{pl}} \varepsilon$, and thus $\delta \subseteq \varepsilon$ by the inductive hypothesis. Since $\gamma \in \text{PlOrd}$, we must have $\delta \in \gamma$. $\square$

Lemma 6.1.17. *For any $\alpha, \beta, \beta', \gamma \in \text{PlOrd}$, if $\gamma \in \alpha$, then $\alpha \cdot^{\text{pl}} \beta \subseteq \alpha \cdot^{\text{pl}} \beta' +^{\text{pl}} \gamma$ implies $\beta \subseteq \beta'$.*

Proof. We show this by set induction on β' . Consider any $\delta \in \beta$, we know that

$$\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \in (\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma)^{\text{pl}+} \subseteq \alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \alpha.$$

Thus $\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \in \alpha \cdot^{\text{pl}} \beta \subseteq \alpha \cdot^{\text{pl}} \beta' +^{\text{pl}} \gamma$.

Now, it suffices to show, by another set induction on γ here, that $\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \in \alpha \cdot^{\text{pl}} \beta' +^{\text{pl}} \gamma$ implies $\delta \in \beta'$ (for the fixed β' above). By the definition of plump ordinal addition, we must have either $\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \in \alpha \cdot^{\text{pl}} \beta'$, or $\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \in (\alpha \cdot^{\text{pl}} \beta' +^{\text{pl}} \gamma')^{\text{pl}+}$ for some $\gamma' \in \gamma$.

In the first case, there must exist $\varepsilon \in \beta'$ such that $\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \in \alpha \cdot^{\text{pl}} \varepsilon +^{\text{pl}} \alpha$, i.e. there exists $\gamma' \in \alpha$ such that $\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \in (\alpha \cdot^{\text{pl}} \varepsilon +^{\text{pl}} \gamma')^{\text{pl}+}$ and hence $\alpha \cdot^{\text{pl}} \delta \subseteq \alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \subseteq \alpha \cdot^{\text{pl}} \varepsilon +^{\text{pl}} \gamma'$. By the inductive hypothesis of the outer induction, $\delta \subseteq \varepsilon$, thus $\delta \in \beta'$ by plumpness.

In the second case, we must have

$$\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma' \in (\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma')^{\text{pl}+} \subseteq \alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \subseteq \alpha \cdot^{\text{pl}} \beta' +^{\text{pl}} \gamma',$$

and thus $\delta \in \beta'$ by the inductive hypothesis of the inner induction. $\square$

Corollary 6.1.18. *For any $\alpha, \beta \in \text{PlOrd}$, the function $f : \alpha \times \beta \rightarrow \alpha \cdot^{\text{pl}} \beta$ given by $\langle \gamma, \delta \rangle \mapsto \alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma$ is injective.*

Proof. We first check that f is indeed well-defined on codomain $\alpha \cdot^{\text{pl}} \beta$: for any

$\gamma \in \alpha, \delta \in \beta$, we see that

$$\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \in (\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma)^{\text{pl}+} \subseteq \alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \alpha \subseteq \alpha \cdot^{\text{pl}} \beta.$$

Now, suppose that $f(\gamma, \delta) = f(\gamma', \delta')$. Observe that we have $\alpha \cdot^{\text{pl}} \delta \subseteq f(\gamma, \delta) = \alpha \cdot^{\text{pl}} \delta' +^{\text{pl}} \gamma'$ and thus $\delta \subseteq \delta'$ by Lemma 6.1.17. Symmetrically, we have $\delta' \subseteq \delta$, i.e. $\delta = \delta'$. This implies $\alpha \cdot^{\text{pl}} \delta = \alpha \cdot^{\text{pl}} \delta'$ and hence furthermore $\gamma = \gamma'$ by Lemma 6.1.16. $\square$

We will first use this construction to define an intuitionistic analogue of the classical result that every L_α is in $(\Sigma_1\text{-definable})$ bijection with an initial segment of ordinals:

Theorem 6.1.19 (IKP($\mathcal{P}$), [50, Theorem 4.5]). *There exist class functions Θ, I on domain PlOrd , such that for any $\alpha \in \text{PlOrd}$, we have $\Theta(\alpha) \in \text{PlOrd}$ and $I(\alpha) \subseteq \Theta(\alpha) \times L_\alpha$ is a surjection from a subset of $\Theta(\alpha)$ onto L_α .*

Proof. We define Θ and I simultaneously by Σ -recursion, using the following process where $\mathcal{P}$ is treated as a primitive symbol (and thus PlOrd is Δ_0). Fix some $\alpha \in \text{PlOrd}$, denote $\gamma = \bigcup_{\beta \in \alpha} \Theta(\beta)$ and let

$$\Theta(\alpha) = \alpha \cdot^{\text{pl}} \omega^{\text{pl}} \cdot^{\text{pl}} \gamma^\omega,$$

where ω^{pl} equals the union of all finite plump successors of $\emptyset$ and γ^ω is the union of all finite plump products $\gamma \cdot^{\text{pl}} \gamma \cdot^{\text{pl}} \dots$. $I(\alpha)$ will precisely contain those $\langle \delta, x \rangle \in \Theta(\alpha) \times L_\alpha$ such that there exists $\beta \in \alpha$, formula φ and $\varepsilon_1, \dots, \varepsilon_n \in \text{dom}(I(\beta)) \subseteq \Theta(\beta)$ such that

- (i) $\delta = \sum_{i=1}^n \alpha \cdot^{\text{pl}} \omega^{\text{pl}} \cdot^{\text{pl}} \gamma^{n-i} \cdot^{\text{pl}} \varepsilon_i +^{\text{pl}} \alpha \cdot^{\text{pl}} q^{\text{pl}} +^{\text{pl}} \beta$, where $q \in \omega$ is the Gödel numbering of φ ;
- (ii) $x = \{z \in L_\beta : L_\beta \models \varphi(z, I(\beta)(\varepsilon_1), \dots, I(\beta)(\varepsilon_n))\} \in \text{def}(L_\beta)$.

This is expressible internally in IKP($\mathcal{P}$) using the standard Gödel numbering and the internal truth predicates for set models presented in section 4.3.

Now, repeated applications of Corollary 6.1.18 shows immediately that if $\langle \delta, x \rangle, \langle \delta', x' \rangle \in I(\alpha)$ with $\delta = \delta'$, then x and x' are defined through exactly the same formula and parameters, thus $I(\alpha)$ is indeed a well-defined function. The other desired properties are all straightforward to verify. $\square$

It is not hard to see that by the remark on page 122, the class functions Θ and

I will be definable within L (or L_{pl}) as well, through a formula which contains an existential quantifier that ranges over the plump ordinals.

On the other hand, we observe that Corollary 6.1.18 additionally allows us to code a pairwise strongly incomparable function between plump ordinals by a single plump ordinal. More specifically:

Proposition 6.1.20. *There is a formula $\varphi(x, y; \alpha, \beta, \sigma)$ such that, for any $\alpha, \beta \in \text{PlOrd}$, if $f : s \rightarrow \beta$ is pairwise strongly incomparable on domain $s \subseteq \alpha$, then there exists some $\sigma \in \text{PlOrd}$ such that*

$$(\gamma \in s \wedge f(\gamma) = \delta) \leftrightarrow \varphi(\gamma, \delta; \alpha, \beta, \sigma).$$

Proof. We take

$$\sigma = \bigcup_{\gamma \in s} (\alpha \cdot^{\text{pl}} f(\gamma) +^{\text{pl}} \gamma)^{\text{pl}+},$$

and define $\varphi(x, y; \alpha, \beta, \sigma)$ as the formula

$$x \in \alpha \wedge y \in \beta \wedge \exists \tau \in \sigma \left(\tau = \alpha \cdot^{\text{pl}} y +^{\text{pl}} x \wedge \forall \tau' \in \sigma \left(\tau \subseteq \tau' \rightarrow \tau = \tau' \right) \right).$$

For the forward direction, we see for any $\gamma \in s$ that indeed $\alpha \cdot^{\text{pl}} f(\gamma) +^{\text{pl}} \gamma \in \sigma$. Additionally, for any $\tau \in \sigma$, there must exist $\gamma' \in s$ such that $\tau \subseteq \alpha \cdot^{\text{pl}} f(\gamma') +^{\text{pl}} \gamma'$. Now, if $\alpha \cdot^{\text{pl}} f(\gamma) +^{\text{pl}} \gamma \subseteq \tau$, i.e.

$$\alpha \cdot^{\text{pl}} f(\gamma) \subseteq \alpha \cdot^{\text{pl}} f(\gamma) +^{\text{pl}} \gamma \subseteq \tau \subseteq \alpha \cdot^{\text{pl}} f(\gamma') +^{\text{pl}} \gamma',$$

then by Lemma 6.1.17, we must also have $f(\gamma) \subseteq f(\gamma')$, which by pairwise strong incomparability implies $\gamma = \gamma'$. It then follows trivially that $\tau = \alpha \cdot^{\text{pl}} f(\gamma) +^{\text{pl}} \gamma$, i.e. $\varphi(\gamma, f(\gamma); \alpha, \beta, \sigma)$ holds.

For the backward direction, assume that $\varphi(\gamma, \delta; \alpha, \beta, \sigma)$ holds. Then we must have $\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \in \sigma$, i.e. $\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma \subseteq \alpha \cdot^{\text{pl}} f(\gamma') +^{\text{pl}} \gamma'$ for some $\gamma' \in s$, which also implies $\alpha \cdot^{\text{pl}} \delta +^{\text{pl}} \gamma = \alpha \cdot^{\text{pl}} f(\gamma') +^{\text{pl}} \gamma'$ by stipulation in φ . Since also $\gamma, \delta \in \text{PlOrd}$ and $\gamma \in \alpha$, it follows from Corollary 6.1.18 that $\delta = f(\gamma')$ and $\gamma = \gamma'$. Thus we have $\gamma \in s \wedge \delta = f(\gamma)$ as desired. $\square$

Since plump ordinals are always in L by Lemma 6.1.13, and plump ordinal arithmetic is definable in L by the remark on page 122, these coded functions must also be in L . We thus have the following:

Theorem 6.1.21 (IKP($\mathcal{P}$), [50, Theorem 4.7]). *Suppose that $s \subseteq \text{PlOrd}$ and*

$f : s \rightarrow \text{PlOrd}$ is pairwise strongly incomparable, then $\mathcal{P}(s) \subseteq L$. More specifically, there exists a plump ordinal $\alpha \in \text{PlOrd}$ such that $\mathcal{P}(s) \subseteq L_\alpha$.

Proof. Let $\alpha = \bigcup_{\beta \in s} \beta^{\text{pl}+}$ so that $s \subseteq \alpha$ and let $x \subseteq s$ be any subset. We consider

$$\sigma_x = \bigcup_{\beta \in x} f(\beta)^{\text{pl}+},$$

then σ_x is a plump ordinal, thus $\sigma_x \in L$ by Lemma 6.1.13. We can then define

$$s_x = \{\beta \in \alpha : \beta \in s \wedge f(\beta) \in \sigma_x\}.$$

By Proposition 6.1.20, the condition $\beta \in s \wedge f(\beta) \in \sigma_x$ can be equivalently written as $\exists \gamma \in \sigma_x \varphi(\beta, \gamma)$ where φ is a Δ_0 -formula except it contains plump ordinal additions and multiplications. Since these operations are definable inside L , we know that $s_x \in L$.

Now obviously, for any $\beta \in x$, we have $f(\beta) \in \sigma_x$, thus $\beta \in s_x$. On the other hand, for any $\beta \in s_x$ we have $\beta \in s$ and some $\gamma \in x$ such that $f(\beta) \in f(\gamma)^{\text{pl}+}$, i.e. $f(\beta) \subseteq f(\gamma)$, thus $\beta = \gamma \in x$ by incomparability. Therefore, we indeed have $x = s_x \in L$.

Now, observe that every ordinal parameter we have used in the process is a plump ordinal, i.e. some $\delta \in \text{def}(L_\delta)$ by Lemma 6.1.13. Thus, by taking plump successors, we can ensure that each s_x lies in some L_{δ_x} where $\delta_x \in \text{PlOrd}$. It follows that $\mathcal{P}(s) \subseteq L_{\bigcup_{x \in \mathcal{P}(s)} \delta_x}$, or simply $\mathcal{P}(s) \subseteq L_{\text{pl}}$ using the aforementioned class notation $L_{\text{pl}} = \bigcup_{\alpha \in \text{PlOrd}} L_\alpha$. $\square$

Corollary 6.1.22 (IKP($\mathcal{P}$)). *Suppose that for any $\alpha \in \text{PlOrd}$, there exists a pairwise strongly incomparable function $f : \alpha \rightarrow \text{PlOrd}$, then we have $V = L$.*

Proof. We prove by set induction that every set x lies in L_{pl} (hence in L), i.e. $x \in L_\beta$ for some plump ordinal β . By Σ_1 -collection, it suffices to show that for any set $x \subseteq L_\alpha$ where $\alpha \in \text{PlOrd}$, we also have $x \in L_\beta$ for some $\beta \in \text{PlOrd}$. Consider the plump ordinal $\Theta(\alpha)$ given in Theorem 6.1.19, then Theorem 6.1.21 implies that $\mathcal{P}(\Theta(\alpha)) \subseteq L_\beta$ for some plump ordinal β . Specifically, the following set

$$\sigma = \{\gamma \in \Theta(\alpha) : \exists y \in x \langle \gamma, y \rangle \in I(\alpha)\} \in L_\beta.$$

However, as we remarked just after Theorem 6.1.19, the class functions Θ and

I are definable in L_{pl} as well. It follows that

$$s = \{I(\alpha)(\gamma) : \gamma \in \sigma\} \in L_{\text{pl}}.$$

For any $y \in x$, there exists $\gamma \in \Theta(\alpha)$ such that $I(\alpha)(\gamma) = y$ by surjectivity and $\gamma \in \sigma$ by its definition, thus $x \subseteq s$. On the other hand, since $I(\alpha)$ is a function, for any $\gamma \in \sigma$, $I(\alpha)(\gamma) \in x$ by definition of σ . Therefore, indeed $x = s \in L_{\text{pl}}$. $\square$

6.2 Weak realisability with powerset

In the remainder of this chapter, we shall present a realisability model of $\text{CZF}(\mathcal{P})$ where we indeed have as many incomparable objects as needed in Corollary 6.1.22. We will work in a classical base theory $\text{KP}(\mathcal{P})$ where it is more convenient to manipulate large sets. We consider a nested realisability implementation, where we first construct a variant of weak realisability in [32] to interpret $\text{CZF}(\mathcal{P})$, and then build within this model the realisability interpretation described in section 4.2 to introduce the incomparable ordinals.

The weak realisability conditions shall use a class-sized pca $E_\varphi(X)$ of *set recursive functions* extended by the powerset operation and an additional (partial) Σ_1 -class function X , where the latter will be used later to supply some incomparable sets from the classical base theory to the intuitionistic models.

Since we will simultaneously interact with two notions of realisability using different pcas, we will distinguish their application terms by underlining the terms in $E_\varphi(X)$ (in this chapter only), i.e. we write $\underline{st}$ for the application of function s on t in $E_\varphi(X)$; we will write terms without underlines for the pca of the inner realisability model — which will be Kleene's first algebra Kl .

Now, $E_\varphi(X)$ will have the entire universe V as its domain and the partial application function will be a class function given by Σ -inductive definition. We pick arbitrary distinct $\mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{s}_{\mathbb{N}}, \mathbf{p}_{\mathbb{N}}, \mathbf{d}_{\mathbb{N}}, \boldsymbol{\gamma}, \boldsymbol{\rho}, \boldsymbol{\nu}, \boldsymbol{\pi}, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, \bar{\varphi}, \mathbf{x} \in \omega$, and first informally specify the following application behaviours in $E_\varphi(X)$:

$$\begin{aligned} \underline{\mathbf{k}xy} \downarrow &= x, \\ \underline{\mathbf{s}xyz} \downarrow &= \underline{xz(yz)} \quad \text{when the right-hand side converges,} \\ \underline{\mathbf{p}xy} \downarrow &= \langle x, y \rangle, \\ \underline{\mathbf{p}_0x} \downarrow &= y \quad \text{when } x = \langle y, z \rangle, \\ \underline{\mathbf{p}_1x} \downarrow &= z \quad \text{when } x = \langle y, z \rangle, \end{aligned}$$

$$\begin{aligned}
\underline{\mathbf{s}}_{\mathbb{N}}n\downarrow &= n + 1 \quad \text{when } n \in \omega, \\
\underline{\mathbf{p}}_{\mathbb{N}}n\downarrow &= m \quad \text{when } n = m + 1, m \in \omega, \\
\underline{\mathbf{d}}_{\mathbb{N}}nmxy\downarrow &= \begin{cases} x & \text{when } n = m \in \omega, \\ y & \text{when } n, m \in \omega, n \neq m, \end{cases} \\
\underline{\gamma}xy\downarrow &= x \cap \left(\bigcap y \right), \\
\underline{\rho}xy\downarrow &= \{ \underline{xu} : u \in y \} \quad \text{when } \underline{xu}\downarrow \text{ for all } u \in y, \\
\underline{\nu}x\downarrow &= \bigcup x, \\
\underline{\pi}xy\downarrow &= \{x, y\}, \\
\underline{\mathbf{i}}_1xyz\downarrow &= \begin{cases} x & \text{when } y \in z, \\ \emptyset & \text{when } y \notin z, \end{cases} \\
\underline{\mathbf{i}}_2xyz\downarrow &= \{u \in x : u \in y \rightarrow u \in z\}, \\
\underline{\mathbf{i}}_3xyz\downarrow &= \{u \in x : u \in y \rightarrow z \in u\}, \\
\underline{\bar{\phi}}x\downarrow &= \mathcal{P}(x), \\
\underline{\mathbf{x}}x\downarrow &= y \quad \text{when } y \text{ is some unique set such that } \langle x, y \rangle \in X,
\end{aligned}$$

for any set x, y, z . Any application term not listed here is expected to diverge.

Formally speaking, the partial application function that satisfies these stipulations will be some inductive definition $I(\Phi)$ where Φ is a Σ_1 -class given as follows, using the same technique as [32]:

For any clause $\underline{ex_1 \cdots x_k}\downarrow = z$ appearing in the listing above, we will have

- (1) if $k > 1$, then $\langle \langle \langle e, x \rangle, \langle e, x \rangle \rangle, \emptyset \rangle \in \Phi$ for any set x , i.e. we stipulate that $\underline{ex}\downarrow = \langle e, x \rangle$;
- (2) for any $1 < i < k$, we have

$$\langle \langle \langle \langle e, x_1, \dots, x_{i-1} \rangle, x_i \rangle, \langle e, x_1, \dots, x_i \rangle \rangle, \emptyset \rangle \in \Phi$$

for any set $x_1, \dots, x_i$, i.e. we stipulate that

$$\underline{\langle e, x_1, \dots, x_{i-1} \rangle x_i}\downarrow = \langle e, x_1, \dots, x_i \rangle;$$

- (3) finally, we have $\langle \langle \langle e, x_1, \dots, x_{k-1}, x_k \rangle, z \rangle, y \rangle \in \Phi$ for any set $x_1, \dots, x_k$, or simply $\langle \langle \langle e, x_1 \rangle, z \rangle, y \rangle$ when $k = 1$, where the computation should converge and y is the set of partial applications that are needed to produce output z .

For example, for $\mathbf{s}$, we need the following entries

$$\begin{aligned} \langle \langle \langle \mathbf{s}, x \rangle, \langle \mathbf{s}, x \rangle \rangle, \emptyset \rangle &\in \Phi, \\ \langle \langle \langle \langle \mathbf{s}, x \rangle, y \rangle, \langle \mathbf{s}, x, y \rangle \rangle, \emptyset \rangle &\in \Phi, \\ \langle \langle \langle \langle \mathbf{s}, x, y \rangle, z \rangle, w \rangle, \{ \langle \langle x, z \rangle, u \rangle, \langle \langle y, z \rangle, v \rangle, \langle \langle u, v \rangle, w \rangle \} \rangle &\in \Phi \end{aligned}$$

for any x, y, z, u, v, w . Since every distinguished operation e is a natural number, every nonempty e must contain $\emptyset$ as an element; on the other hand, ordered pairs and tuples implemented in the Kuratowski style are nonempty sets that do not contain $\emptyset$ as an element, thus every distinguished operation e is distinct from every ordered pair, and it is easy to verify that $I(\Phi)$ is indeed a partial function $V^2 \rightharpoonup V$ (cf. the proof that the inductive definition $\mathbf{U}_A$ is a function in Lemma 5.3.1).

Now, it is easy to see that $E_\varphi(X)$ is a pca over natural numbers by explicit stipulation. We can then define an interpreted realisability interpretation on it where the variables also range over the domain of all sets V . However, the pca cannot compute choice functions and thus cannot ‘pick an arbitrary element’ from a nonempty set⁴², which will cause problems (for example, in Lemma 6.2.5 later).

Therefore, we follow [32] and consider *weak realisability* conditions, usually denoted as $\Vdash_{\mathbf{w}}$, which use modified conditions for the disjunction and existential clauses that essentially allow one to specify a nonempty set of possible realisers without picking a specific one.

To begin, the atomic clauses are trivial:

$$\begin{aligned} d \Vdash_{\mathbf{w}} x \in y & \iff x \in y, \\ d \Vdash_{\mathbf{w}} x = y & \iff x = y. \end{aligned}$$

The conjunction, negation, implication and universal clauses still follow section 2.3, whereas the disjunction and existential clauses are as follows:

$$\begin{aligned} d \Vdash_{\mathbf{w}} \varphi \vee \psi & \iff d \neq \emptyset \wedge \forall e \in d \left((\underline{\mathbf{p}}_0 e = 0 \wedge \underline{\mathbf{p}}_1 e \downarrow \Vdash_{\mathbf{w}} \varphi) \right. \\ & \quad \left. \vee (\underline{\mathbf{p}}_0 e = 1 \wedge \underline{\mathbf{p}}_1 e \downarrow \Vdash_{\mathbf{w}} \psi) \right), \\ d \Vdash_{\mathbf{w}} \exists x \varphi & \iff d \neq \emptyset \wedge \forall e \in d \left(\underline{\mathbf{p}}_0 e \downarrow \wedge \underline{\mathbf{p}}_1 e \downarrow \Vdash_{\mathbf{w}} \varphi[\underline{\mathbf{p}}_0 e / x] \right). \end{aligned}$$

⁴²Admittedly, our construction contains a custom Σ_1 -class function, which could encode a choice function when such a function is definable in the base theory, e.g. when $V = L$ holds. However, since weak realisability itself is an interesting notion, we will present it here precisely as given in [32].

Finally, the conditions for bounded quantifiers are

$$\begin{aligned} d \Vdash_{\mathfrak{w}} \forall x \in b \varphi & : \Longleftrightarrow \quad \forall c \in b \, \underline{dc} \downarrow \Vdash_{\mathfrak{w}} \varphi[c/x], \\ d \Vdash_{\mathfrak{w}} \exists x \in b \varphi & : \Longleftrightarrow \quad d \neq \emptyset \wedge \forall e \in d \left(\underline{\mathbf{p}_0 e} \downarrow \in b \wedge \underline{\mathbf{p}_1 e} \downarrow \Vdash_{\mathfrak{w}} \varphi[\underline{\mathbf{p}_0 e}/x] \right). \end{aligned}$$

Here, the realisability model shares the same domain as the base theory, thus we will give it a new name $V_{E_\varphi(X)}$. We write $V_{E_\varphi(X)} \models \varphi$ to abbreviate for $\exists d \, d \Vdash_{\mathfrak{w}} \varphi$, where φ is a first-order formula possibly containing any set as parameters.

We now verify that $V_{E_\varphi(X)}$ is a realisability model for the theory $\text{CZF}(\mathcal{P})$. While some realisability conditions have changed, we observe that most logical axioms and rules can still be verified in the same way as in Proposition 2.3.1, as long as we observe that singleton sets can be constructed trivially as an unordered pair

$$\mathfrak{sg} = \underline{\lambda x. \pi x x},$$

so that $\underline{\mathfrak{sg} x} \downarrow = \{x\}$. However, the axiom (8) of disjunction elimination and the inference rule of $\exists$ -elimination will require some additional treatment, using the following technical lemma:

Lemma 6.2.1 ([32, Lemma 3.2]). *For any formula $\varphi(\vec{x})$ in the language of set theory with all free variables indicated, there exists a term χ_φ in the pca $E_\varphi(X)$ such that $\text{KP}(\mathcal{P})$ proves*

$$\forall \vec{x} \forall c \left(c \neq \emptyset \wedge (\forall d \in c \, d \Vdash_{\mathfrak{w}} \varphi(\vec{x})) \rightarrow \underline{\chi_\varphi \vec{x} c} \downarrow \Vdash_{\mathfrak{w}} \varphi(\vec{x}) \right).$$

Proof. We construct χ_φ by induction on the complexity of φ : for the atomic and negation cases, the actual realiser is irrelevant, so we can simply take $\chi_\varphi = \underline{\lambda \vec{x} c. \emptyset}$.

For the conjunction, implication and universal cases, we just defer the construction to some χ_ψ of a subformula ψ . Explicitly,

$$\begin{aligned} \chi_{\varphi \wedge \psi} &= \underline{\lambda \vec{x} c. \mathbf{p}(\chi_\varphi \vec{x}(\underline{\mathbf{p} \mathbf{p}_0 c}))(\chi_\psi \vec{x}(\underline{\mathbf{p} \mathbf{p}_1 c}))}, \\ \chi_{\varphi \rightarrow \psi} &= \underline{\lambda \vec{x} c d. \chi_\psi \vec{x}(\underline{\mathbf{p}(\lambda f. f d)} c)}, \\ \chi_{\forall y \varphi(y)} &= \chi_{\forall y \in t \varphi(y)} = \underline{\lambda \vec{x} c y. \chi_\varphi \vec{x} y(\underline{\mathbf{p}(\lambda f. f y)} c)}, \end{aligned}$$

where in the bounded universal quantifier case, t denotes any variable from $\vec{x}$.

Finally, for the disjunction and existential cases, observe that a realiser is

simply defined to be a ‘set of realisers’ by stipulation, and thus it suffices to simply take $\chi_\varphi = \underline{\lambda \vec{x}c. \nu c}$. $\square$

Corollary 6.2.2. *Any logical axioms or inference rules are realised in $V_{E_\varphi(X)}$.*

Proof. The only non-trivial axiom is disjunction elimination

$$(\varphi \rightarrow \sigma) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\varphi \vee \psi \rightarrow \sigma)),$$

realised by $\underline{\lambda \vec{x} f g c. \chi_\sigma \vec{x}(\rho(\lambda d. \mathbf{d}_\mathbb{N}(\mathbf{p}_0 d) 0 f g(\mathbf{p}_1 d)) c)}$ where we use χ_σ to deal with the set of realisers for $\varphi \vee \psi$.

Likewise, for the $\exists$ -elimination rule, assume that $f \Vdash_{\mathbf{w}} \varphi \rightarrow \psi$ where the free variables are $x, \vec{y}$ and x does not occur free in ψ , then $\exists x \varphi \rightarrow \psi$ is realised by $\underline{\lambda \vec{y} c. \chi_\psi \vec{y}(\rho(\lambda d. f(\mathbf{p}_0 d) \vec{y}(\mathbf{p}_1 d)) c)}$. $\square$

Now, equality is realised trivially here, so the equality axioms are already covered by Lemma 2.3.2. For the axioms for bounded quantifiers:

Lemma 6.2.3. *The bounded quantifier axioms are realised in $V_{E_\varphi(X)}$:*

$$\begin{aligned} \forall x (x \in a \rightarrow \varphi(x)) &\leftrightarrow \forall x \in a \varphi(x), \\ \exists x (x \in a \wedge \varphi(x)) &\leftrightarrow \exists x \in a \varphi(x). \end{aligned}$$

Proof. For any formula $\varphi(x)$ with no other free variables (but possibly set parameters), we simply observe that

$$\begin{aligned} (d \Vdash_{\mathbf{w}} \forall x \in b \varphi(x)) &\leftrightarrow (\underline{\lambda x e. d x} \Vdash_{\mathbf{w}} \forall x (x \in b \rightarrow \varphi(x))), \\ (d \Vdash_{\mathbf{w}} \exists x \in b \varphi(x)) &\leftrightarrow \{\langle x, \langle \emptyset, e \rangle \rangle : \langle x, e \rangle \in d\} \Vdash_{\mathbf{w}} \exists x (x \in b \wedge \varphi(x)). \end{aligned}$$

For the former, $\underline{\lambda d x e. d x}$ simply computes a realiser for the right-hand side from the left-hand side, whereas $\underline{\lambda d x. d x \emptyset}$ computes the converse.

For the latter, $\underline{\rho(\lambda e. \mathbf{p}(\mathbf{p}_0 e)(\mathbf{p} \emptyset(\mathbf{p}_1 e)))}$ will then compute a realiser for the right-hand side from the left-hand side, whereas for the converse computation, $\underline{\rho(\lambda e. \mathbf{p}(\mathbf{p}_0 e)(\mathbf{p}_{11} e))}$ will suffice. $\square$

For the non-logical axioms, we shall begin by proving a variant of [32, Proposition 3.7]. This is a technical result that will be quite useful when reasoning with the axioms:

Proposition 6.2.4. *For any Δ_0 -formula $\varphi(\vec{x})$ with all free variables indicated, possibly containing the primitive symbol $\mathcal{P}$, there exist terms d_φ, k_φ in the pca $E_\varphi(X)$ such that $\text{KP}(\mathcal{P})$ proves, for every tuple $\vec{x} \in V^k$ (of appropriate, finite length),*

- (i) $\underline{d_\varphi \vec{x}} \downarrow \in \{0, 1\}$ and $\underline{d_\varphi \vec{x}} = 1$ if and only if $\varphi(\vec{x})$ holds;
- (ii) if $\underline{d_\varphi \vec{x}} = 1$, then $\underline{k_\varphi \vec{x}} \downarrow \Vdash_{\mathfrak{w}} \varphi(\vec{x})$; if $\underline{d_\varphi \vec{x}} = 0$, then $\neg \exists e \Vdash_{\mathfrak{w}} \varphi(\vec{x})$.

It follows immediately that $\text{KP}(\mathcal{P})$ proves that $\varphi(\vec{x})$ holds if and only if $\exists e \Vdash_{\mathfrak{w}} \varphi(\vec{x})$.

Proof. We construct d_φ, k_φ by induction on the complexity of φ :

- (1) When φ is the atomic formula $s \in t$, where s and t are either a variable or a finite number of applications of the symbol $\mathcal{P}$ to a variable, it suffices to take $d_\varphi = \underline{\lambda \vec{x}. \mathbf{i}_1 1 s' t'}$, where $\vec{x}$ enumerates the free variables of φ and s', t' are corresponding (formal) terms in $E_\varphi(X)$ by replacing occurrences of $\mathcal{P}$ in s and t with $\bar{\varphi}$ instead; when φ is the formula $s = t$, we observe that it suffices to simply take

$$d_\varphi = d_{\forall z \in s \, z \in t \wedge \forall z \in t \, z \in s}$$

by extensionality in the base theory, which can be handled later in the bounded quantifier case. In either case, we just need $k_\varphi = \underline{\lambda \vec{x}. \emptyset}$.

- (2) When φ is of the form $\psi \wedge \chi$, it suffices to take⁴³

$$d_\varphi = \underline{\lambda \vec{x}. \gamma 1(\pi(d_\psi \vec{x})(d_\chi \vec{x}))},$$

which enforces $\underline{d_\varphi \vec{x}} = \underline{d_\psi \vec{x}} \cap \underline{d_\chi \vec{x}}$, and simply $k_\varphi = \underline{\lambda \vec{x}. \mathbf{p}(k_\psi \vec{x})(k_\chi \vec{x})}$.

- (3) When φ is of the form $\psi \vee \chi$, it suffices to take

$$d_\varphi = \underline{\lambda \vec{x}. \nu(\pi(d_\psi \vec{x})(d_\chi \vec{x}))},$$

i.e. $\underline{d_\varphi \vec{x}} = \underline{d_\psi \vec{x}} \cup \underline{d_\chi \vec{x}}$, and

$$k_\varphi = \underline{\lambda \vec{x}. \nu(\pi(\mathbf{d}_{\mathbb{N}}^*(d_\psi \vec{x}) 1(\mathbf{sg}(\mathbf{p} 0(k_\psi \vec{x}))) \emptyset)(\mathbf{d}_{\mathbb{N}}^*(d_\chi \vec{x}) 1(\mathbf{sg}(\mathbf{p} 1(k_\chi \vec{x}))) \emptyset))},$$

⁴³It is trivial to resolve the differences between how free variables are enumerated for ψ and χ . So we can assume without loss of generality that d_ψ, k_ψ , etc. all take the same inputs as d_φ, k_φ that we are constructing here. Same applies to other cases below.

where $\mathbf{sg} = \lambda x. \pi x x$ is the singleton constructor, and $\underline{k_\varphi \vec{x}}$ evaluates to

$$\left\{ \left\langle 0, \underline{k_\psi \vec{x}} \right\rangle : \underline{d_\psi \vec{x}} = 1 \right\} \cup \left\{ \left\langle 1, \underline{k_\chi \vec{x}} \right\rangle : \underline{d_\chi \vec{x}} = 1 \right\}.$$

- (4) When φ is of the form $\neg\psi$, we can simply handle it in the implication case below by taking $d_\varphi = d_{\psi \rightarrow \perp}$, where $\perp$ is a new propositional symbol with no possible realisers and $d_\perp = k_\perp = \emptyset$; clearly $k_\varphi = \underline{\lambda \vec{x}. \emptyset}$ works.
- (5) When φ is of the form $\psi \rightarrow \chi$, observe that we just have to take $\underline{d_\varphi \vec{x}} = \left\{ \emptyset : \emptyset \in \underline{d_\psi \vec{x}} \rightarrow \emptyset \in \underline{d_\chi \vec{x}} \right\}$, i.e.

$$d_\varphi = \underline{\lambda \vec{x}. \mathbf{i}_2 1(d_\psi \vec{x})(d_\chi \vec{x})}.$$

By the inductive hypothesis, suppose that $\psi(\vec{x}) \rightarrow \chi(\vec{x})$ is true and we have any realiser $e \Vdash_{\mathbf{w}} \psi(\vec{x})$, then $\underline{k_\chi \vec{x}} \Vdash_{\mathbf{w}} \chi(\vec{x})$; so for k_φ all we need is

$$k_\varphi = \underline{\lambda \vec{x} e. k_\chi \vec{x}}.$$

- (6) When φ is of the form $\forall y \in t \psi(y)$ where t is either a variable or a finite number of applications of the symbol $\mathcal{P}$ to a variable, it suffices to take

$$\begin{aligned} d_\varphi &= \underline{\lambda \vec{x}. \gamma 1(\rho(\lambda y. d_\psi \vec{x} y) t')}, \\ k_\varphi &= \underline{\lambda \vec{x} y. k_\psi \vec{x} y}, \end{aligned}$$

where t' is the (formal) term in $E_\varphi(X)$ by replacing occurrences of $\mathcal{P}$ in t with $\bar{\varphi}$ instead.

- (7) When φ is of the form $\exists y \in t \psi(y)$ where t is either a variable or a finite number of applications of the symbol $\mathcal{P}$ to a variable, we take

$$\begin{aligned} d_\varphi &= \underline{\lambda \vec{x}. \nu(\rho(\lambda y. d_\psi \vec{x} y) t')}, \\ k_\varphi &= \underline{\lambda \vec{x}. \nu(\rho(\lambda y. \mathbf{d}_\mathbb{N}^*(d_\psi \vec{x} y) 1(\mathbf{sg}(\mathbf{p} y(k_\psi \vec{x} y))) \emptyset) t')}, \end{aligned}$$

where $\underline{k_\varphi \vec{x}}$ evaluates to

$$\bigcup_{y \in t} \left\{ \left\langle y, \underline{k_\psi \vec{x} y} \right\rangle : \underline{d_\psi \vec{x} y} = 1 \right\}.$$

□

It follows that any Π_1 -formula has a uniform realiser for all variable assignments on which it is true in the base theory. Therefore, for example, we

immediately know that the Π_1 -formula stating the axiom of extensionality is realised simply by virtue of it being true in the base theory. Likewise, we know that the unordered pairs and unions are computable by π, ν in the pca, and it follows that we can uniformly realise the Δ_0 -statements ‘ πxy is the unordered pair of x and y ’ or ‘ νx is the union of x ’ in $V_{E_\varphi(X)}$. The axiom of infinity (read as the Δ_0 -claim that ω is an inductive set), the binary intersection axiom and the powerset axiom are likewise trivial.

The case for the set induction scheme is completely analogous to Lemma 4.2.8, so all that remains is the strong collection axiom:

Lemma 6.2.5. *The strong collection scheme is realised in $V_{E_\varphi(X)}$:*

$$\forall x \in a \exists y \theta(x, y) \rightarrow \exists b (\forall x \in a \exists y \in b \theta(x, y) \wedge \forall y \in b \exists x \in a \theta(x, y)).$$

Proof. Let $\vec{z}$ enumerate the free variables in $\theta(x, y)$ other than x, y and a . For any set $a, \vec{c}$ and $f \Vdash_{\mathfrak{w}} \forall x \in a \exists y \theta(x, y)[\vec{c}/\vec{z}]$, we can define

$$g = \lambda a f. \nu(\rho(\lambda x. \rho \mathbf{p}_0(fx))a),$$

then for any $x \in a$, $\underline{fx} \Vdash_{\mathfrak{w}} \exists y \in b \theta(x, y)[\underline{gaf}/b, \vec{c}/\vec{z}]$. Conversely, for any $y \in \underline{gaf}$, we compute

$$h = \lambda a f y. \nu(\rho(\lambda x. \nu(\rho(\lambda d. \mathbf{d}_{\mathbb{N}}(d_{x=y}y(\mathbf{p}_0d))1(\mathbf{sg}(\mathbf{p}x(\mathbf{p}_1d)))\emptyset)(fx)))a),$$

where we use $d_{x=y}$ from Proposition 6.2.4 to select those entries in $\underline{fx}$ where the first component is y , i.e. we compute

$$\underline{hafy} = \bigcup_{x \in a} \{ \langle x, \mathbf{p}_1d \rangle : d \in \underline{fx}, \mathbf{p}_0d = y \},$$

so that for any $y \in \underline{gaf}$, $\underline{hafy} \Vdash_{\mathfrak{w}} \exists x \in a \theta(x, y)[\vec{c}/\vec{z}]$. It follows that

$$\underline{\lambda a \vec{b} f. \mathbf{sg}(\mathbf{p}(gaf)(\mathbf{p}f(haf)))}$$

realises the axiom. □

By Proposition 4.1.2, subset collection is implied by powerset and hence does not need to be realised separately. Thus, we have so far verified that:

Theorem 6.2.6. *Let φ be any theorem of $\text{CZF}(\mathcal{P})$, then for any Σ_1 -class function X , $\text{KP}(\mathcal{P})$ proves $V_{E_\varphi(X)} \models \varphi$, i.e. there exists $d \in E_\varphi(X)$ such that*

$d \Vdash_{\mathbf{w}} \varphi$.

Now, we will remark that one may consider a weaker pca $E(X)$ already inside KP (or even IKP), which omits the powerset operation $\bar{\varphi}$. Let CZF^- denote the subtheory CZF – subset collection (which is sometimes also denoted CZF^0 , e.g. in [30]), then KP or IKP would suffice to verify that all axioms of CZF^- are realised (cf. Rathjen [33]). However, we cannot prove that subset collection is realised there, because even for the axiom of exponentiation, Exp , the statement claiming ‘ f is the set of all functions between x and y ’ is Π_1 . Therefore, being able to compute a realiser for the existence of such f from arbitrary x, y would imply, by Proposition 6.2.4, that f is indeed the set of all functions between x and y in the base theory, which KP does not suffice to prove.

Additionally, observe that Proposition 6.2.4 implies the following:

Lemma 6.2.7. *$V_{E_\varphi(X)}$ realises the axiom scheme of restricted excluded middle, REM,*

$$\varphi \vee \neg\varphi$$

for any Δ_0 -formula φ .

Proof. Let $\vec{x}$ enumerate the free variables in φ , then

$$\lambda\vec{x}. \mathbf{sg}(\mathbf{p}(1 - d_\varphi\vec{x})(\mathbf{d}_\mathbb{N}(d_\varphi\vec{x})1k_\varphi k_{\neg\varphi}\vec{x}))$$

simply realises the axiom. □

Since REM implies the linearity of ordinals, etc., $V_{E_\varphi(X)}$ is not a ‘very intuitionistic model’, and it will not contain any incomparable ordinals. Thus, in the next section, we will need to consider another realisability interpretation nested in this model of $\text{CZF}(\mathcal{P}) + \text{REM}$ to continue our construction.

We will additionally remark here that it is not hard to make $V_{E_\varphi(X)}$ itself realise $V = L$. Simply observe that the statement $V = L$ is a Π_2 -formula, and we can write it in some form

$$\forall x \exists y \psi(x, y)$$

where ψ is Δ_0 (such that this formula and the original formulation of $V = L$ are provably equivalent even in IKP) and it is provable in IKP that for every x , the corresponding y satisfying $\psi(x, y)$, if it exists, is unique. Now assume that the base theory satisfies $\text{KP}(\mathcal{P}) + V = L$, then it follows that $x \mapsto y$ given by this formula is a total Δ_0 -class function, so we can build $V_{E_\varphi(X)}$ where X

is precisely this function. Now, by Proposition 6.2.4, we then have

$$\underline{k_\psi x(\mathbf{x}x)} \Vdash_{\mathbf{w}} \psi(x, \underline{\mathbf{x}x}),$$

and we have that $\underline{\lambda x. \mathbf{sg}(\mathbf{p}(\mathbf{x}x)(k_\psi x(\mathbf{x}x)))}$ realises $V = L$ for $V_{E_\varphi(X)}$.

However, this is an interpretation of $\text{CZF}(\mathcal{P}) + \text{REM} + V = L$ inside $\text{KP}(\mathcal{P}) + V = L$, where both theories are much stronger than $\text{KP}(\mathcal{P})$ or $\text{CZF}(\mathcal{P})$, as we have discussed in section 5.2. We will instead present a construction in a weaker base theory in the next section.

6.3 An equiconsistency proof for $\text{CZF}(\mathcal{P}) + V = L$

We now consider the realisability model $V(\mathcal{A})$ defined in section 4.2. We first additionally see that:

Lemma 6.3.1. *$\text{CZF}(\mathcal{P})$ proves that the powerset axiom is realised in $V(\mathcal{A})$:*

$$\forall a \exists y \forall x (x \in y \leftrightarrow x \subseteq a).$$

Proof. Given any $a \in V(\mathcal{A})$, we consider

$$\tilde{a} = \{\langle e, z \rangle : \langle d, z \rangle \in a, e \in \mathcal{A}\}$$

and let $y = \{\langle f, u \rangle : u \in \mathcal{P}(\tilde{a}), f \in \mathcal{A}, f \Vdash u \subseteq a\} \in V(\mathcal{A})$. Note that for fixed u, a , the formula $f \Vdash u \subseteq a$ is essentially

$$\forall \langle d, x \rangle \in u \, f d \downarrow \in F_\in(x, a)$$

using $F_\in$ from section 4.2, and thus a Δ_0 -formula. Therefore, y is indeed a set by Δ_0 -separation and collection.

It now suffices to come up with realisers $t_1, t_2 \in \mathcal{A}$ such that

$$t_1 \Vdash \forall x \in y \, x \subseteq a, \quad t_2 \Vdash \forall x (x \subseteq a \rightarrow x \in y).$$

The first direction is trivial, as $t_1 = \lambda f. f$ simply realises $\forall x \in y \, x \subseteq a$ by the definition of y .

To compute t_2 , assume some $f \Vdash x \subseteq a$. By strong collection, let z be a set

such that

$$\begin{aligned} & \forall \langle d, u \rangle \in x \exists \langle d, v \rangle \in z (\langle \mathbf{p}_0(fd), v \rangle \in a \wedge \mathbf{p}_1(fd) \Vdash u = v) \\ & \wedge \forall \langle d, v \rangle \in z \exists \langle d, u \rangle \in x (\langle \mathbf{p}_0(fd), v \rangle \in a \wedge \mathbf{p}_1(fd) \Vdash u = v). \end{aligned}$$

Clearly $g_1 = \mathbf{p}(\lambda x. \mathbf{p}x(\mathbf{p}_1(fx)))(\lambda x. \mathbf{p}x(\iota_t(\mathbf{p}_1(fx)))) \Vdash x = z$. We also have $g_2 = \lambda x. \mathbf{p}(\mathbf{p}_0(fx))\iota_r \Vdash z \subseteq a$, thus $\langle g_2, z \rangle \in y$ and it follows immediately that $\mathbf{p}g_2g_1 \Vdash x \in y$. $\square$

Specifically, in this section we will focus only on the pca $\mathcal{A} = \text{Kl}$, Kleene's first algebra. Observe that its partial application function is definable by an arithmetic formula, i.e. a Δ_0 -formula in the language of set theory. Thus by Proposition 6.2.4, we know that both the classical base universe V and the weak realisability interpretation $V_{E_\varphi(\text{Kl})}$ will agree on which set defines this pca. Likewise, we can define the collections of names

$$V(\text{Kl})_\alpha = \bigcup_{\beta \in \alpha} \mathcal{P}(\omega \times V(\text{Kl})_\beta),$$

which is now a hierarchy of sets and can be constructed using Σ -recursion instead of inductive definitions, since we have the powerset axiom. We take $V(\text{Kl}) = \bigcup_{\alpha \in \text{Ord}} V(\text{Kl})_\alpha$ and it is trivial to check (in any extension of $\text{IKP}(\mathcal{P})$) that this coincides with the construction of $V(\mathcal{A}) = I(\Phi)$ in section 4.2, by verifying that the class $V(\text{Kl})$ satisfies both requirements in Theorem 4.2.1 using set induction on Ord . We will additionally see that:

Lemma 6.3.2. *There is a term $t \in E_\varphi(X)$ such that $x \in V(\text{Kl})$ if and only if $tx \downarrow \Vdash_w x \in V(\text{Kl})$.*

Proof. Here $V(\text{Kl})$ is a class, so the claim is not trivial. However, since we phrased the construction of $V(\text{Kl})$ through Σ -recursion, we see that the statement ' $z = V(\text{Kl})_\alpha$ ' can be explicitly expressed as 'there exists $f : \alpha^+ \rightarrow V$ such that each $f(\beta) = \bigcup_{\gamma \in \beta} \mathcal{P}(\omega \times f(\gamma))$ and $f(\alpha) = z$ ', where the part inside the existential quantifier is a Δ_0 -formula containing the powerset symbol $\mathcal{P}$.

Now, we can compute the function $\alpha \mapsto V(\text{Kl})_\alpha$ in $E_\varphi(X)$ by recursion as

$$\underline{\mathbf{r}(\lambda \alpha f. \nu(\rho(\lambda \beta. \bar{\varphi}(\nu(\rho(\lambda n. \rho(\mathbf{p}n)(f\beta))\omega)))\alpha))},$$

so it is also easy to compute the (set-theoretic) witness function f above. By Proposition 6.2.4, V and $V_{E_\varphi(X)}$ will agree on which f is the witness to

$z = V(\text{Kl})_\alpha$, and thus they will agree on which set is $V(\text{Kl})_\alpha$ as well for every ordinal α .

Therefore, it suffices to show that, given any $x \in V(\text{Kl})$, $E_\varphi(X)$ can compute some α such that $x \in V(\text{Kl})_\alpha$. However, we just need to compute its rank function,

$$\text{rk}(x) = \sup \{ \text{rk}(y)^+ : \langle n, y \rangle \in x \},$$

which is simply

$$\underline{\mathbf{r}(\lambda x f. \nu(\rho(\lambda y. \nu(\pi(f(\mathbf{p}_1 y))(\mathbf{s}g(f(\mathbf{p}_1 y))))x))}. \quad \square$$

From now onwards, we will read $V(\text{Kl}) \models \varphi$ as the claim in the base theory that

$$\exists e e \Vdash_{\mathbf{w}} \exists n \in \text{Kl } n \Vdash \varphi.$$

The formula φ can contain parameters from $V(\text{Kl})$, since by Lemma 6.3.2 the base theory knows correctly which sets are names inside $V_{E_\varphi(X)}$.

A strongly incomparable ω -sequence

The first task towards $V = L$ now is to recover the pairwise strongly incomparable function on ω that we constructed in section 5.4, which will be implemented here as a sequence of names $S_0 : \omega \rightarrow V(\text{Kl})$ inside the base theory $\text{KP}(\mathcal{P})$. As enabled by the theory of plump ordinals, which we will use later, we do not care about a uniform presentation $\alpha_0 \in V(\text{Kl})$ and hence do not need the specific pairing function. However, we will still work with finite ω -tuples and the operations $\mathbf{l}_s, \mathbf{t}_s \in \text{Kl}$. For a function $f : \omega \rightarrow \omega$, we shall write $f \upharpoonright n \in \omega$ to denote (the code for) the initial segment $\langle f(0), \dots, f(n-1) \rangle$.

Proposition 6.3.3. *There exists a family of functions $\langle f_n : \omega \rightarrow \omega \rangle_{n \in \omega}$ such that for any $i, j, e \in \omega$, if $i \neq j$ then there exists some $m \in \omega$ satisfying*

$$e(f_i \upharpoonright m) \downarrow \rightarrow e(f_i \upharpoonright m) \neq f_j \upharpoonright m.$$

Proof. Similar to Proposition 5.4.3, we construct the family f_n as an increasing sequence of finite objects $t^{(0)}, t^{(1)}, t^{(2)}, \dots$ coded as natural numbers. Here we will pick each $t^{(i)}$ as a finite tuple of finite tuples of natural numbers, i.e. $t^{(i)} = \langle t_0^{(i)}, \dots, t_{k-1}^{(i)} \rangle \in (\omega^{<\omega})^{<\omega}$, and require that for $i < j$, we have $\mathbf{l}_s t^{(i)} \leq \mathbf{l}_s t^{(j)}$ and $t_k^{(i)} \sqsubseteq t_k^{(j)}$ for each $0 \leq k < \mathbf{l}_s t^{(i)}$.

Specifically, we will let $t^{(0)}$ be the empty tuple, and for each i such that $\mathbf{p}_{00}i \neq$

$\mathbf{p}_{01}i$, we will first extend $t_{\mathbf{p}_{00}i}^{(i)}$ to an extension s strictly longer than $\mathbf{l}_s t_{\mathbf{p}_{01}i}^{(i)}$ (adding empty tuples to $t^{(i)}$ if it does not yet contain a $(\mathbf{p}_{00}i)^{\text{th}}$ or $(\mathbf{p}_{01}i)^{\text{th}}$ component), and then extend $t_{\mathbf{p}_{01}i}^{(i)}$ to an s' such that $\mathbf{l}_s s' = \mathbf{l}_s s$ and $\mathbf{p}_1 i s$ does not compute s' correctly. We should build $t^{(i+1)}$ by replacing the $(\mathbf{p}_{00}i)^{\text{th}}$ and $(\mathbf{p}_{01}i)^{\text{th}}$ components in $t^{(i)}$ with s and s' respectively.

In order to ensure that the limit is a family of functions on domain ω , we additionally require that for each i , $\mathbf{l}_s t^{(i)} \geq i$ and also $\mathbf{l}_s t_k^{(i)} \geq i$ for each $0 \leq k < i$, by filling in extra 0's whenever necessary. Since each $t^{(i)}$ is a finite object, we can again choose the one satisfying the requirements above with the least code at each step. By Δ_0 -separation, we can then define each function f_n such that $f_n(k)$ is the k^{th} component of $t_n^{(m)}$ where $m = \max\{n, k\} + 1$. $\square$

Now, in the base theory $\text{KP}(\mathcal{P})$, we shall say that a function $f : x \rightarrow V(\text{Kl})$ is a *sequence of pairwise strongly incomparable names* if for any $u, v \in x$,

$$(V(\text{Kl}) \models f(u) \subseteq f(v)) \rightarrow u = v.$$

We see it follows from the construction above that:

Corollary 6.3.4. *There is a sequence of pairwise strongly incomparable names $S_0 : \omega \rightarrow V(\text{Kl})$ such that for any $n \in \omega$, $V(\text{Kl}) \models S_0(n) \in \mathcal{P}(\omega) \cap \text{Ord}$.*

Proof. Take the family of functions $\langle f_n \rangle_{n \in \omega}$ as in Proposition 6.3.3. For any $n \in \omega$, we let

$$S_0(n) = \{ \langle f_n \restriction m, \overline{m} \rangle : m \in \omega \} \in V(\text{Kl})$$

where $\overline{m} = \{ \langle k, \overline{k} \rangle : k \in m \} \in V(\text{Kl})$ are the same recursively defined standard names for natural numbers introduced in section 4.3, and likewise

$$\overline{\omega} = \{ \langle m, \overline{m} \rangle : m \in \omega \}$$

is a standard name for ω in $V(\text{Kl})$.

Now, Turing computability, the properties we asserted for f_n in Proposition 6.3.3 and the definition of S_0 are all expressible as Δ_0 -formulae, so by Proposition 6.2.4, $V_{E_\varphi(X)}$ knows that they satisfy the same properties. We can thus work with these sets inside $V_{E_\varphi(X)}$, a model of $\text{CZF}(\mathcal{P}) + \text{REM}$.

Clearly we see $\lambda t. \mathbf{p}(\mathbf{l}_s t)_{\iota_r} \Vdash S_0(n) \subseteq \overline{\omega}$, and likewise, $\lambda t m. \mathbf{p}(\mathbf{t}_s t m)_{\iota_r} \Vdash$

$S_0(n)$ is transitive. Reasoning inside $V(\text{Kl})$, it follows immediately that also

$$V(\text{Kl}) \models S_0(n) \in \text{Ord}.$$

For strong incomparability, since the base theory $\text{KP}(\mathcal{P})$ is classical, it suffices to assume $i \neq j \in \omega$ and show that $V_{E_\varphi(X)}$ realises $\neg \exists e \in \text{Kl } e \Vdash S_0(i) \subseteq S_0(j)$. Working inside $V_{E_\varphi(X)}$ which knows $i \neq j$, suppose that some $e \Vdash S_0(i) \subseteq S_0(j)$. Now, by induction on ω , we know that for $m, n \in \omega$, if any $d \Vdash \bar{m} = \bar{n}$, then $m = n$ must be true. It follows that for any $m \in \omega$, we must have $\mathbf{p}_0(e(f_i \upharpoonright m)) \downarrow = f_j \upharpoonright m$. However, this contradicts the requirement in Proposition 6.3.3 that any function in Kl , specifically $\lambda t. \mathbf{p}_0(et)$ here, cannot compute $f_j \upharpoonright m$ correctly from $f_i \upharpoonright m$ for all $m \in \omega$. $\square$

Afterwards, we will describe an iterated process to build the specific partial Σ_1 -class function X we need in the base theory $\text{KP}(\mathcal{P})$, which starts from S_0 and combines plump mechanisms inside $V(\text{Kl})$ and techniques from classical set theory in $\text{KP}(\mathcal{P})$. Thus we shall first say a bit about how one can conveniently extract complicated constructions inside $V(\text{Kl})$ to play with in the base theory.

Since our realisability structure is nested inside the model of weak realisability $V_{E_\varphi(X)}$, we will need to add some choice principles to the base theory to extract a single, canonical name from the existential realisers. We will consider the extension of $\text{KP}(\mathcal{P})$ by the *axiom of global choice*, $\text{AC}_{\text{global}}$ in [36], which adds a new binary relation symbol R and the new axioms

$$\begin{aligned} \forall x \forall y \forall z (R(x, y) \wedge R(x, z) \rightarrow y = z), \\ \forall x (x \neq \emptyset \rightarrow \exists y \in x R(x, y)), \end{aligned}$$

specifying that R is a choice class function. The symbol R is also allowed to appear in Δ_0 -formulae in axiom schemes such as Δ_0 -separation or Δ_0 -collection. It follows that $\{\langle x, y \rangle : R(x, y)\}$ is a Δ_0 -class function (containing the primitive symbol R) that computes an element in x for every nonempty set x .

We observe that, whenever $\text{CZF}(\mathcal{P})$ proves a formula φ of any complexity, then $\text{CZF}(\mathcal{P})$ also proves that this formula is realised by some $f \in \text{Kl}$, and it follows from Theorem 6.2.6 that there is a realiser $h \in E_\varphi(X)$ for

$$\exists f \in \text{Kl } f \Vdash \varphi$$

or any provably equivalent forms. Since $E_\varphi(X)$ is inductively defined, h and

derived application terms can be expressed as partial Σ_1 -class functions, possibly involving the primitive powerset symbol $\mathcal{P}$.

Additionally, since the realisers for axioms of $\text{CZF}(\mathcal{P})$ never use the specific operation $\mathbf{x}$ in $E_\varphi(X)$, it is easy to verify that the values of h remain the same regardless of which class function we use as X — most notably, we can define the same h in $E_\varphi(\emptyset)$, and it will be unproblematic to, conversely, construct the class X using Σ_1 -class functions like h .

For example, in section 6.1 we defined the plump operator $(-)^{\text{pl}} : \text{Ord} \rightarrow \text{PlOrd}$ in $\text{IKP}(\mathcal{P})$ and observe that, using Lemma 6.1.9 and Lemma 6.1.10, it is very easy to generalise it to any set x such that

$$x^{\text{pl}} = \bigcup_{y \in x} (y^{\text{pl}})^{\text{pl}+}$$

and verify that still $x^{\text{pl}} \in \text{PlOrd}$. In other words, we have shown

$$\text{CZF}(\mathcal{P}) \vdash \forall x \exists y (y = x^{\text{pl}} \wedge y \in \text{PlOrd}),$$

and it follows that, by expanding the definition of realisability conditions for $V_{E_\varphi(X)}$, there is some $h \in E_\varphi(\emptyset)$ that realises

$$\exists d \in \omega \forall x (x \in V(\text{Kl}) \rightarrow \exists y (y \in V(\text{Kl}) \wedge d \Vdash y = x^{\text{pl}} \wedge y \in \text{PlOrd})).$$

Let h' be any element in h . For any $x \in V(\text{Kl})$ in the base theory $\text{KP}(\mathcal{P})$, we have the term t in Lemma 6.3.2 which computes a realiser for $x \in V(\text{Kl})$. Using global choice, we can define e_x as some unique element (in the realiser for the inner existential subformula) such that

$$R(\langle \mathbf{p}_1 h' \rangle x(tx), e_x)$$

holds, then we will have $\mathbf{p}_0 e_x \in V(\text{Kl})$ and $V(\text{Kl}) \models \mathbf{p}_0 e_x \in \text{PlOrd}$. We will abuse notation and denote this Σ_1 -class function (containing the primitive symbol R) on $V(\text{Kl})$ given by $x \mapsto \mathbf{p}_0 e_x$ also as $(-)^{\text{pl}}$. Notice that e_x additionally computes realisers for $\mathbf{p}_0 e_x = x^{\text{pl}}$ being plump, or more precisely:

Lemma 6.3.5 ($\text{KP}(\mathcal{P}) + \text{AC}_{\text{global}}$). *There is some $d_0 \in \omega$ such that for any function $f : x \rightarrow V(\text{Kl})$, let $g : x \rightarrow V(\text{Kl})$ be given by $g(z) = f(z)^{\text{pl}}$, then there is $\mathbf{p} \in E_\varphi(X)$ such that*

$$\mathbf{p} \Vdash_w \forall z \in x d_0 \Vdash g(z) \in \text{PlOrd}.$$

Proof. Consider the realiser h for

$$\exists d \in \omega \forall x (x \in V(\text{Kl}) \rightarrow \exists y (y \in V(\text{Kl}) \wedge d \Vdash y = x^{\text{pl}} \wedge y \in \text{PlOrd})) .$$

and fixed $h' \in h$ as on the previous page. Observe that the desired $d_0 \in \omega$ is then simply $\mathbf{p}_1(\mathbf{p}_0 h')$ — note that the inner unpairing function is in $E_\varphi(X)$, whereas the outer one is in Kl .

On the other hand, note that $e : x \rightarrow E_\varphi(\emptyset)$ given by $z \mapsto \mathbf{p}_1 e_z$, where e_z satisfies $R(\mathbf{p}_1 h' z(tz), e_z)$, is a set-theoretic function by strong Σ_1 -collection. Here $\mathbf{p}_1 e_z$ realises the statement

$$\mathbf{p}_0 h' \Vdash g(z) = f(z)^{\text{pl}} \wedge g(z) \in \text{PlOrd}$$

in $V_{E_\varphi(X)}$. Because $E_\varphi(X)$ can evaluate set-theoretic functions⁴⁴, $\mathbf{p}$ just need to take $z \in x$ and extract the needed information from the set parameter e . $\square$

Corollary 6.3.6. *There is a sequence of pairwise strongly incomparable names $T_0 : \omega \rightarrow V(\text{Kl})$ such that $V_{E_\varphi(X)} \models \forall n \in \omega d_0 \Vdash T_0(n) \in \text{PlOrd}$.*

The natural choice is to let $T_0(n) = S_0(n)^{\text{pl}}$ using S_0 from Corollary 6.3.4. We need a lemma:

Lemma 6.3.7 (IKP($\mathcal{P}$)). *If α is a linear ordinal, i.e. it satisfies the trichotomy*

$$\forall \beta, \gamma \in \alpha (\beta \in \gamma \vee \beta = \gamma \vee \gamma \in \beta) ,$$

such as ω , then for any $\beta, \gamma \in \mathcal{P}(\alpha) \cap \text{Ord}$, we have

$$\beta^{\text{pl}} \subseteq \gamma^{\text{pl}} \rightarrow \beta \subseteq \gamma .$$

Proof. Assume $\beta^{\text{pl}} \subseteq \gamma^{\text{pl}}$ and consider any $\delta \in \beta$. By definition, we have $\delta^{\text{pl}} \in \gamma^{\text{pl}}$, i.e. $\delta^{\text{pl}} \in (\varepsilon^{\text{pl}})^{\text{pl}+}$ for some $\varepsilon \in \gamma$. Observe that $\varepsilon \in \delta$ implies $\varepsilon^{\text{pl}} \in \delta^{\text{pl}}$, which implies $\varepsilon^{\text{pl}} \in \delta^{\text{pl}} \subseteq \varepsilon^{\text{pl}}$ and thus a contradiction. By trichotomy, we must then have $\delta \in \varepsilon$ or $\delta = \varepsilon$. In either case, we have $\delta \in \gamma$ as desired. $\square$

Proof of Corollary 6.3.6. Take S_0 in Corollary 6.3.4 and let

$$T_0(n) = S_0(n)^{\text{pl}} .$$

⁴⁴Specifically, we have $e_z = e(z) = \mathbf{p}_1(\nu(\gamma(\rho(\lambda t. \mathbf{p}z(\mathbf{p}_1 t))e)(\pi ee)))$.

Observe that Lemma 6.3.5 gives the realisers for plumpness, so it suffices to check strong incomparability.

For any $i, j \in \omega$, suppose that $T_0(i) \subseteq T_0(j)$ is realised. We know that $V(\text{Kl}) \models T_0(i) = S_0(i)^{\text{pl}} \wedge T_0(j) = S_0(j)^{\text{pl}}$, and it recognises additionally that $S_0(i), S_0(j)$ are subsets of ω , where ω is a linear ordinal by induction on it. It follows from Lemma 6.3.7 that

$$V(\text{Kl}) \models T_0(i) \subseteq T_0(j) \rightarrow S_0(i) \subseteq S_0(j),$$

thus $V(\text{Kl}) \models S_0(i) \subseteq S_0(j)$ and $i = j$ by assumptions in Corollary 6.3.4. $\square$

Classical incomparables

The iterated process we shall now describe will use T_0 as a starting point to build sequences of pairwise strongly incomparable names on arbitrarily large domains x .

For this process to work over the entire universe, first note that in [36], Rathjen modified the relativised ordinal analysis of $\text{KP}(\mathcal{P})$ in [35] to prove that $\text{KP}(\mathcal{P}) + \text{AC}_{\text{global}} \equiv_{\text{Con}} \text{KP}(\mathcal{P})$. The precise proof-theoretic treatment will not be relevant to this thesis, but we will observe that, by Lemma 5.2.2, the theory

$$\text{KP}(\mathcal{P}) + \text{AC}_{\text{global}} + \omega_1 \text{ exists}$$

proves $\text{Con}(\text{KP}(\mathcal{P}))$ and thus $\text{Con}(\text{KP}(\mathcal{P}) + \text{AC}_{\text{global}})$ as well. It follows that, assuming $\text{KP}(\mathcal{P}) + \text{AC}_{\text{global}}$ is consistent, then it cannot prove ω_1 exists and thus

$$\text{KP}(\mathcal{P}) + \text{AC}_{\text{global}} + \text{every } \alpha \in \text{Ord} \text{ is countable}$$

is consistent. In other words, this extension is equiconsistent with $\text{KP}(\mathcal{P})$ itself. We will now work in this extended base theory for the remainder of the section, and look at the following constructions:

Lemma 6.3.8. *There exists some $f \in E_\varphi(\emptyset)$ such that, if $T : x \rightarrow V(\text{Kl})$ is a function in the base theory and $V_{E_\varphi(X)} \models \forall y \in x \, d_0 \Vdash T(y) \in \text{PlOrd}$, then $\underline{fT} \in V(\text{Kl})$ is the ‘union’ of elements in the image of T , in the sense that $V(\text{Kl}) \models \underline{fT} \in \text{PlOrd}$ and, for any $z \in V(\text{Kl})$, $V(\text{Kl}) \models z \in \underline{fT}$ if and only if $V(\text{Kl}) \models z \in T(y)$ for some $y \in x$.*

Proof. What we need is simply $\underline{fT} = \bigcup_{y \in x} T(y)$. Reasoning inside $V_{E_\varphi(X)}$, it

is easy to derive that for all $d \in \text{Kl}$, $z \in V(\text{Kl})$,

$$d \Vdash z \in \underline{fT} \leftrightarrow \exists y \in x \, d \Vdash z \in T(y).$$

Now, observe that this additionally means

$$\mathbf{p}(\mathbf{p0})\mathbf{p}_1 \Vdash \forall z \left(z \in \underline{fT} \leftrightarrow \exists y \in T^* \, z \in y \right),$$

where T^* is the name $\{\langle 0, T(y) \rangle : y \in x\} \in V(\text{Kl})$, i.e. $V(\text{Kl}) \models \underline{fT} = \bigcup T^*$. Since by assumptions, we know $V_{E_\varphi(X)} \models \lambda y. d_0 \Vdash \forall y \in T^* \, y \in \text{PIOrd}$, thus we can apply Lemma 6.1.10 inside $V(\text{Kl})$ to derive that $V(\text{Kl}) \models \underline{fT} \in \text{PIOrd}$. $\square$

Lemma 6.3.9. *Suppose that a, T, e are set-theoretic functions on domain $\alpha \in \text{Ord}$ and for each $\beta \in \alpha$, $T(\beta) : a(\beta) \rightarrow V(\text{Kl})$ is a sequence of pairwise strongly incomparable names, $e(\beta) \in E_\varphi(\emptyset)$ and $e(\beta) \Vdash_{\mathfrak{w}} \forall x \in a(\beta) \, d_0 \Vdash T(\beta)(x) \in \text{PIOrd}$, then there is some $T' : \bigcup_{\beta \in \alpha} a(\beta) \rightarrow V(\text{Kl})$ that is also pairwise strongly incomparable, and*

$$V_{E_\varphi(X)} \models \forall x \in \bigcup_{\beta \in \alpha} a(\beta) \, d_0 \Vdash T'(x) \in \text{PIOrd}.$$

Additionally, this T' (and the relevant realiser, which must lie in $E_\varphi(\emptyset)$) can be uniquely determined from a, T, e using a Σ_1 -formula containing the primitive symbols $\mathcal{P}$ and R .

Proof. Our extended base theory claims that $\alpha \in \text{Ord}$ must be countable, so we can uniquely find some injection $\rho : \alpha \rightarrow \omega$ such that $R(u, \rho)$ holds where u is the set of all injections from α to ω .

Now, first observe that e is a set-theoretic function and, as $E_\varphi(X)$ can evaluate set-theoretic functions (cf. footnote 44), $V_{E_\varphi(X)} \models \forall \beta \in \alpha \, \forall x \in a(\beta) \, d_0 \Vdash T(\beta)(x) \in \text{PIOrd}$. Similar to how we previously extracted the plump operator $(-)^{\text{pl}}$ into the base theory, we know that the plump successor $(-)^{\text{pl}+}$, addition $+^{\text{pl}}$ and multiplication $\cdot^{\text{pl}}$ are also computed by some terms in $E_\varphi(X)$, taking as input elements in $V(\text{Kl})$ and the realisers that they are plump (consisting of entries from both $E_\varphi(X)$ and Kl). Consequently, we can compute, for example,

$$\eta = \bigcup_{\beta \in \alpha, x \in a(\beta)} T(\beta)(x)^{\text{pl}+},$$

and by Lemma 6.3.8, $V(\text{Kl}) \models \eta \in \text{PIOrd}$. We can then define $S' : \bigcup_{\beta \in \alpha} a(\beta) \rightarrow$

$V(\text{Kl})$ as

$$S'(x) = \eta \cdot^{\text{pl}} T_0(\rho(\beta_x)) +^{\text{pl}} T(\beta_x)(x),$$

where $\beta_x = \min \{\beta \in \alpha : x \in a(\beta)\}$ and T_0 is as given in Corollary 6.3.6.

It is easy to verify that, for any $x \in \bigcup_{\beta \in \alpha} a(\beta)$, $V(\text{Kl}) \models S'(x) \in \text{PlOrd}$ by applying results in section 6.1 inside $V(\text{Kl})$. To see that S' is pairwise strongly incomparable, suppose that $V(\text{Kl}) \models S'(x) \subseteq S'(y)$. Inside $V(\text{Kl})$, we know that $T(\beta_x)(x), T(\beta_y)(y) \in \eta$, thus by Lemma 6.1.17, we must have $T_0(\rho(\beta_x)) \subseteq T_0(\rho(\beta_y))$. This implies $\rho(\beta_x) = \rho(\beta_y)$ in the base theory and thus $\beta_x = \beta_y$ by injectivity. It then follows from Lemma 6.1.16 inside $V(\text{Kl})$ that $T(\beta_x)(x) \subseteq T(\beta_y)(y)$, and thus $x = y$.

Now, observe that the plump ordinals generated by ordinal arithmetic do not guarantee that their plumpness can be realised by the uniform realiser d_0 . Thus, we additionally set $T'(x) = S'(x)^{\text{pl}}$, so that they have the correct realisers by Lemma 6.3.5. Notice that for each $x \in \bigcup_{\beta \in \alpha} a(\beta)$, $V(\text{Kl})$ does know $S'(x) \in \text{PlOrd}$ (although by a non-uniform realiser), so $V(\text{Kl}) \models T'(x) = S'(x)$ by Proposition 6.1.12. It is thus easy to verify that also $V(\text{Kl}) \models T'(x) \subseteq T'(y)$ implies $x = y$. $\square$

Corollary 6.3.10. *Suppose that $T : x \rightarrow V(\text{Kl})$ is a sequence of pairwise strongly incomparable names and $V_{E_\varphi(X)} \models \forall y \in x \, d_0 \Vdash T(y) \in \text{PlOrd}$, then there is some $T' : x \times 2 \rightarrow V(\text{Kl})$ (which can be determined using a Σ_1 -formula), that is also pairwise strongly incomparable and realised to be a collection of plump ordinals.*

Proof. Simply use Lemma 6.3.9 where $\alpha = 2$, $a(i) = \{\langle y, i \rangle : y \in x\}$, on some function

$$S(i)(\langle y, i \rangle) = T(y). \quad \square$$

Lemma 6.3.11. *Suppose that $T : x \rightarrow V(\text{Kl})$ is a sequence of pairwise strongly incomparable names and $V_{E_\varphi(X)} \models \forall y \in x \, d_0 \Vdash T(y) \in \text{PlOrd}$, then there is some $T' : \mathcal{P}(x) \rightarrow V(\text{Kl})$ (which can be determined using a Σ_1 -formula), that is also pairwise strongly incomparable and uniformly realised to be a collection of plump ordinals (by $d_0 \in \omega$).*

Proof. Since we are working in a classical base theory, we have a function $\chi : \mathcal{P}(x) \rightarrow \mathcal{P}(x \times 2)$ given by

$$\chi(s) = \{\langle y, 0 \rangle : y \in s\} \cup \{\langle y, 1 \rangle : y \in x \setminus s\}.$$

It is easy to check that χ is a ‘classical strongly incomparable map’, in the sense that $\chi(s) \subseteq \chi(t) \rightarrow s = t$, simply because each $\chi(s)$ must be a function on domain x .

Now, we let S be the sequence of pairwise strongly incomparable names on $x \times 2$ given by Corollary 6.3.10, and define some $S' : \mathcal{P}(x) \rightarrow V(\text{Kl})$ as

$$S'(s) = \bigcup_{y \in \chi(s)} S(y)^{\text{pl}+},$$

such that $V(\text{Kl}) \models S'(s) \in \text{PIOrd}$ for all $s \subseteq x$ by Lemma 6.3.8. For strong incomparability, suppose that $V(\text{Kl}) \models S'(s) \subseteq S'(t)$. For any $y \in \chi(s)$, we have $V(\text{Kl}) \models S(y) \in S'(s)$ and thus there must exist some $z \in \chi(t)$ such that

$$V(\text{Kl}) \models S(y) \in S(z)^{\text{pl}+},$$

i.e. $V(\text{Kl}) \models S(y) \subseteq S(z)$. It follows from the strong incomparability of S that $y = z$ and thus $\chi(s) \subseteq \chi(t)$, which implies $s = t$ immediately.

Finally, using the same trick as in the proof of Lemma 6.3.9 and taking $T'(s) = S'(s)^{\text{pl}}$ will ensure that the names have uniform realisers for plumpness. $\square$

Now, we consider the powerset hierarchy

$$\begin{aligned} V_0 &= \emptyset, \\ V_{\alpha+} &= \mathcal{P}(V_\alpha), \\ V_\lambda &= \bigcup_{\alpha \in \lambda} V_\alpha \quad \text{for limit } \lambda \end{aligned}$$

inside the classical base theory, and we define two class functions e, T on Ord by Σ -recursion (using formulae containing symbols $\mathcal{P}$ and R):

- (1) Let $k : \omega \rightarrow V_\omega$ be an arbitrary bijection — for example, the Ackermann encoding. Then $T(\omega) : V_\omega \rightarrow V(\text{Kl})$ is given by

$$T(\omega)(x) = T_0(k^{-1}(x))$$

with T_0 as in Corollary 6.3.6 and $e(\omega) \Vdash_{\mathfrak{w}} \forall x \in V_\omega d_0 \Vdash T(\omega)(x) \in \text{PIOrd}$. Additionally, $T(n) = T(\omega)|_{V_n}$ and $e(n) = e(\omega)$ for $n \in \omega$.

- (2) When $\alpha > \omega$ and $\alpha = \beta^+$ is a successor, $T(\alpha) : \mathcal{P}(V_\beta) \rightarrow V(\text{Kl})$ and the corresponding $e(\alpha) \Vdash_{\mathfrak{w}} \forall x \in \mathcal{P}(V_\beta) d_0 \Vdash T(\alpha)(x) \in \text{PIOrd}$ are uniquely determined from $T(\beta)$ and $e(\beta)$ using Lemma 6.3.11.

- (3) When $\alpha > \omega$ is a limit, then $V_\alpha = \bigcup_{\beta \in \alpha} V_\beta$ and $T(\alpha) : V_\alpha \rightarrow V(\text{Kl})$ and the corresponding $e(\alpha) \Vdash_{\mathbf{w}} \forall x \in V_\alpha d_0 \Vdash T(\alpha)(x) \in \text{PlOrd}$ are uniquely determined from $T(\beta)$'s and $e(\beta)$'s using Lemma 6.3.9 and the choice function R .

From here onwards, we fix X as the specific Σ_1 -class function on $V(\text{Kl})$, such that for any $x \in V(\text{Kl})$, we fix $\alpha = \min \{\beta \in \text{Ord} : x \subseteq V_\beta\}$, and define $X(x) = \langle f_x, e(\alpha) \rangle$ where

$$f_x = \{ \langle n, \langle y, T(\alpha)(\langle n, y \rangle) \rangle^\bullet \rangle : \langle n, y \rangle \in x \} \in V(\text{Kl}),$$

where $\langle a, b \rangle^\bullet = \{ \{a\}^\bullet, \{a, b\}^\bullet \}^\bullet$ and $\{a, b\}^\bullet = \{ \langle 0, a \rangle, \langle 1, b \rangle \}$ are the standard names for ordered and unordered pairs in $V(\text{Kl})$. We check that:

Lemma 6.3.12. *$V(\text{Kl})$ realises that for every x , there exists some u such that:*

- (i) $\forall \langle y, \alpha \rangle \in u (y \in x \wedge \alpha \in \text{PlOrd})$;
- (ii) $\forall y \in x \exists \alpha \langle y, \alpha \rangle \in u$;
- (iii) $\forall \langle y, \alpha \rangle, \langle z, \beta \rangle \in u (\alpha \subseteq \beta \rightarrow y = z \wedge \alpha = \beta)$.

Proof. The realisers for $V(\text{Kl})$ just have to realise the three statements separately, but the realisers for $V_{E_\varphi(X)}$ must output u and appropriate additional information given input $x \in V(\text{Kl})$, which is why we need the specific class function X . Fix any input $x \in V(\text{Kl})$, we will simply assign to u the value $f_x = \underline{\mathbf{p}_0(\mathbf{x}x)}$.

Now, the realiser for (i) in $V(\text{Kl})$ is simply⁴⁵ $g = \lambda n. \mathbf{p}(\mathbf{p}n\iota_r)d_0$ using the constant d_0 from Lemma 6.3.5. To see that

$$V_{E_\varphi(X)} \models g \Vdash \forall \langle y, \alpha \rangle \in u (y \in x \wedge \alpha \in \text{PlOrd}),$$

we observe that $V_{E_\varphi(X)}$ knows about the definition of f_x , and for any $\langle n, y \rangle \in x$, the fact that $\mathbf{p}n\iota_r \Vdash y \in x$ is a theorem in $\text{CZF}(\mathcal{P})$ and thus must be realised. To realise the second conjunct, we check that $E_\varphi(X)$ is capable of recovering each $\langle n, y \rangle \in x$ from the corresponding element in f_x since each $\{a, b\}^\bullet$ is essentially a function on $\{0, 1\}$ in the base theory. For any $t = \langle n, \langle y, \alpha \rangle^\bullet \rangle \in f_x$, we just need to apply the $E_\varphi(X)$ -function $e = \underline{\mathbf{p}_1(\mathbf{x}x)}$ to

$$\langle n, y \rangle = \langle n, \underline{\mathbf{p}_1}t(0)(0) \rangle,$$

⁴⁵Here the routine component of a realiser for the defining formula of an ordered tuple is omitted.

which can be easily evaluated in $E_\varphi(X)$, to obtain a realiser of $d_0 \Vdash \alpha \in \text{PIOrd}$.

The realiser for (ii) is trivial, both for the part inside $V(\text{Kl})$ and the part inside $V_{E_\varphi(X)}$, as it is equivalently a simple Δ_0 -claim. For (iii), we claim that the realiser in $V(\text{Kl})$ is simply $\lambda n m d. \mathbf{p}\iota_r \iota_r$. Suppose that given some $\langle n, \langle y, \alpha \rangle^\bullet \rangle, \langle m, \langle z, \beta \rangle^\bullet \rangle \in f_x$, $V(\text{Kl})$ sees a realiser for $\alpha \subseteq \beta$, then this is a realiser for $T(\gamma)(\langle n, y \rangle) \subseteq T(\gamma)(\langle m, z \rangle)$ for some classical $\gamma \in \text{Ord}$, and by our specific, recursive construction of T , this implies $\langle n, y \rangle = \langle m, z \rangle$ and thus also $y = z \wedge \alpha = \beta$ in the classical base theory. When these atomic statements are true, they are also trivially realised in $V_{E_\varphi(X)}$, thus $\underline{\lambda t s d. \mathbf{p}00}$ realises⁴⁶

$$\forall \langle n, \langle y, \alpha \rangle^\bullet \rangle, \langle m, \langle z, \beta \rangle^\bullet \rangle \in f_x ((\exists d \in \text{Kl } d \Vdash \alpha \subseteq \beta) \rightarrow y = z \wedge \alpha = \beta),$$

and working inside $V_{E_\varphi(X)}$, it follows immediately that $\mathbf{p}\iota_r \iota_r$ realises the two equalities in $V(\text{Kl})$. $\square$

We can now work entirely within $V(\text{Kl})$ to show that a system satisfying $\text{CZF}(\mathcal{P})$ and the conditions in Lemma 6.3.12 must satisfy $V = L$. This will involve very similar constructions as done in the proofs of Proposition 6.1.20 and Theorem 6.1.21, but we are unfortunately unable to use the results directly here, since in Lemma 6.3.12 we cannot construct these objects u as functions on x , but only as some kinds of ‘multi-valued functions’.

Theorem 6.3.13. $V(\text{Kl}) \models V = L$.

Proof. Looking at the proof of Corollary 6.1.22, we see that all we need to do is to replace the use of Theorem 6.1.21, i.e. verify that for any $\alpha \in \text{PIOrd}$, also $\mathcal{P}(\alpha) \subseteq L_{\text{pl}}$.

Now, fix any $\alpha \in \text{PIOrd}$. Let u be the corresponding subset of $\alpha \times \text{PIOrd}$ as given by Lemma 6.3.12. We first check that $u \in L_{\text{pl}}$ by imitating Proposition 6.1.20. We define

$$\sigma = \bigcup_{\langle \beta, \gamma \rangle \in u} (\alpha \cdot^{\text{pl}} \gamma +^{\text{pl}} \beta)^{\text{pl}+}, \quad \tau = \bigcup_{\langle \beta, \gamma \rangle \in u} \gamma^{\text{pl}+}.$$

Clearly $\sigma, \tau \in \text{PIOrd}$, thus $\sigma, \tau \in L_{\text{pl}}$. We now extract u from α, σ, τ by defining

$$v = \{ \langle \beta, \gamma \rangle : \beta \in \alpha, \gamma \in \tau, \\ \exists \delta \in \sigma (\delta = \alpha \cdot^{\text{pl}} \gamma +^{\text{pl}} \beta \wedge \forall \delta' \in \sigma (\delta \subseteq \delta' \rightarrow \delta = \delta')) \}.$$

⁴⁶Again, we omitted the process of treating the tuple notations as abbreviations and adjusting the realiser accordingly here.

For any $\langle \beta, \gamma \rangle \in u$, simply let $\delta = \alpha \cdot^{\text{pl}} \gamma +^{\text{pl}} \beta$. Observe that if $\delta \subseteq \delta' \in \sigma$ then there is $\langle \varepsilon, \zeta \rangle \in u$ such that $\delta' \subseteq \alpha \cdot^{\text{pl}} \zeta +^{\text{pl}} \varepsilon$. By Lemma 6.1.17, this implies $\gamma \subseteq \zeta$, which then implies $\beta = \varepsilon$ and $\gamma = \zeta$ by Lemma 6.3.12(iii).

Conversely, for any $\langle \beta, \gamma \rangle \in v$, there must exist $\langle \varepsilon, \zeta \rangle \in u$ such that $\alpha \cdot^{\text{pl}} \gamma +^{\text{pl}} \beta \subseteq \alpha \cdot^{\text{pl}} \zeta +^{\text{pl}} \varepsilon$. Then $\alpha \cdot^{\text{pl}} \zeta +^{\text{pl}} \varepsilon \in \sigma$ and stipulations in the definition of v imply that

$$\alpha \cdot^{\text{pl}} \gamma +^{\text{pl}} \beta = \alpha \cdot^{\text{pl}} \zeta +^{\text{pl}} \varepsilon.$$

It follows that $\beta = \varepsilon$ and $\gamma = \zeta$ by Lemma 6.1.17 and Lemma 6.1.16. Therefore, we have $u = v \in L_{\text{pl}}$ as desired.

Now, consider any subset $x \subseteq \alpha$, and we define $\rho = \bigcup_{\langle \beta, \gamma \rangle \in u, \beta \in x} \gamma^{\text{pl}+}$. Clearly $\rho \in \text{PlOrd}$ and thus $\rho \in L_{\text{pl}}$. Since $u \in L_{\text{pl}}$, we also have

$$y = \{\beta \in \alpha : \exists \gamma \in \rho \langle \beta, \gamma \rangle \in u\} \in L_{\text{pl}}.$$

Obviously $x \subseteq y$ by Lemma 6.3.12(ii). On the other hand, for any $\beta \in y$, there is $\gamma \in \rho$, i.e. $\langle \varepsilon, \zeta \rangle \in u$ satisfying $\varepsilon \in x$, $\gamma \subseteq \zeta$, and also $\langle \beta, \gamma \rangle \in u$. By Lemma 6.3.12(iii) we must have $\beta = \varepsilon$ and $\gamma = \zeta$, thus $\beta \in x$. We have now shown that $x = y \in L_{\text{pl}}$, i.e. $\mathcal{P}(\alpha) \subseteq L_{\text{pl}}$. $\square$

Corollary 6.3.14. *The consistency of $\text{KP}(\mathcal{P})$ implies that of $\text{CZF}(\mathcal{P}) + V = L$. It follows that*

$$\text{CZF}(\mathcal{P}) + V = L \equiv_{\text{Con}} \text{KP}(\mathcal{P}) \equiv_{\text{Con}} \text{CZF}(\mathcal{P}).$$

Since $\text{IKP}(\mathcal{P})$ is a subtheory of $\text{CZF}(\mathcal{P})$ of the same strength, the same equiconsistency also extends easily to $\text{IKP}(\mathcal{P})$ and $\text{IKP}(\mathcal{P}) + V = L$.

Proof. This follows immediately from Theorem 6.3.13. $\square$

Chapter 7

Realisability with large pcas

Observe that the key constructions in chapter 5 and chapter 6 both use the specific pca K1, which means that we have to start from a countable collection of incomparable objects, and even with the iterated construction in section 6.3, we cannot obtain arbitrarily large incomparable collections without some constraints on the height of the universe. In contrast, in [50, section 5], I built large collections of strongly incomparable ordinals directly by forcing on a large Heyting algebra and proved that:

Theorem (ZFC, [50, Corollary 5.9]). *Fix any set x , then there exists a Heyting algebra H such that the corresponding Heyting-valued model V^H satisfies*

$$V^H \models \mathcal{P}(\check{x}) \in L.$$

In this final chapter, we will discuss how the use of larger pcas can allow us to develop similar constructions in realisability interpretations.

7.1 Classical embeddings

We would like to start with a classical system, just as in the forcing scenario, and embed fragments of the base theory into our realisability model. However, this is often inappropriate when studying weak set theories because following Theorem 4.1.3, a classical theory with an unrestricted strong collection principle like in CZF may be much stronger than the intuitionistic counterpart which we would like to model. On the other hand, we may not want to perform interpretations in the form of section 4.2 directly in weaker classical theories like KP, since we do need principles like unrestricted strong collection

to hold in the base theory in order for them to be realised (cf. the proof of Lemma 4.2.9).

Instead, in this chapter we will work inside the base theory $\text{CZF}^- + \text{REM}$ where $\text{CZF}^- = \text{CZF} - \text{subset collection}$, and REM denotes restricted excluded middle, i.e. $\varphi \vee \neg\varphi$ for any Δ_0 -formula φ . As we have remarked in section 6.2, one can always build a model of $\text{CZF}^- + \text{REM}$ from the universe of a model of KP , by interpreting truth in the model through weak realisability conditions.

The construction of the inductive Σ_1 -class $V(\mathcal{A})$ and the corresponding realisability conditions in section 4.2 only requires IKP , so they will work inside CZF^- already. Observe that none of the proofs in that section require subset collection — except verifying the realisability of subset collection itself — thus we immediately have that:

Theorem 7.1.1. *Let φ be any theorem of CZF^- , then CZF^- proves $V(\mathcal{A}) \models \varphi$, i.e. there exists $d \in \mathcal{A}$ such that $d \Vdash \varphi$.*

Now, internally in any universe of intuitionistic set theory, we shall say that a set model $\langle x; \in \rangle$, where x is any transitive set, is Δ_0 -decidable if it satisfies the restricted law of excluded middle, i.e.

$$\forall y_1, \dots, y_n \in x (\varphi \vee \neg\varphi)$$

holds for every Δ_0 -formula φ where $y_1, \dots, y_n$ enumerate all free variables in φ . For any ordinal $\alpha \in \text{Ord}$, we shall say that α *embeds decidable* into the universe if L_α is a Δ_0 -decidable set model.

Lemma 7.1.2 (IKP). *If α embeds decidable into the universe, then for any $\beta \in \alpha^+$, β is a linear ordinal and we have*

$$\forall \gamma \in \beta \gamma^+ = \text{def}(L_\gamma) \cap \text{Ord},$$

i.e. $\beta = L_\beta \cap \text{Ord}$ immediately.

Proof. We prove this by set induction on β . Fix $\beta \in \alpha^+$, then by the inductive hypothesis, every $\gamma \in \beta$ is linear, and

$$\gamma = L_\gamma \cap \text{Ord} \in L_\alpha.$$

We first check that, for any $\delta \in L_\alpha \cap \text{Ord}$, if $\delta \subseteq \gamma$ then either $\delta = \gamma$ or $\delta \in \gamma$. This is completely analogous to the proof of Corollary 4.3.5: by L_α

being Δ_0 -decidable, we have

$$\exists \varepsilon \in \gamma \varepsilon \notin \delta \vee \forall \varepsilon \in \gamma \varepsilon \in \delta.$$

The latter disjunct implies $\delta = \gamma$; in the former case, we show that $\delta \subseteq \varepsilon$. Suppose that this is not the case, i.e. there is some $\eta \in \delta$ such that $\eta \notin \varepsilon$. However, both $\varepsilon, \eta \in \gamma$, thus by the linearity of γ , we have either $\varepsilon \in \eta$ or $\varepsilon = \eta$, both contradicting $\varepsilon \notin \delta$. Thus we must have $\delta \subseteq \varepsilon$, since we can prove Δ_0 -formulae by contradiction inside L_α .

Hence, $\delta \subseteq \gamma$ implies either $\delta = \gamma$ or $\delta \subseteq \varepsilon$ for some $\varepsilon \in \gamma$. By set induction on $\gamma \in \beta$, it follows that

$$\forall \delta \in \text{def}(L_\alpha) \cap \text{Ord} \forall \gamma \in \beta (\delta \subseteq \gamma \rightarrow \delta \in \gamma \vee \delta = \gamma).$$

Thus $\text{def}(L_\gamma) \cap \text{Ord} = \gamma^+$ for every $\gamma \in \beta$, and $\beta = L_\beta \cap \text{Ord}$.

Finally, for linearity, we observe that the argument above can also be applied to certain $\delta \in \text{def}(L_\alpha) \cap \text{Ord}$, where we think of δ as a Δ_0 -definable ‘proper class’ in the Δ_0 -decidable set model L_α . Specifically, we take any $\gamma_1, \gamma_2 \in \beta$ and consider⁴⁷ $\gamma_1 \cap \gamma_2 \in \text{def}(L_\alpha)$. The decidability of L_α then implies

$$\exists \varepsilon \in \gamma' \varepsilon \notin \gamma_1 \cap \gamma_2 \vee \forall \varepsilon \in \gamma' \varepsilon \in \gamma_1 \cap \gamma_2,$$

for any $\gamma' \in \beta$, where $\varepsilon \in \gamma_1 \cap \gamma_2$ can be read as $\varepsilon \in \gamma_1 \wedge \varepsilon \in \gamma_2$ to not involve the set $\gamma_1 \cap \gamma_2$ explicitly. By set induction on γ' , the same argument as above then implies both $\gamma_1 \cap \gamma_2 \in \gamma_1 \vee \gamma_1 \cap \gamma_2 = \gamma_1$ and $\gamma_1 \cap \gamma_2 \in \gamma_2 \vee \gamma_1 \cap \gamma_2 = \gamma_2$. Since having $\gamma_1 \cap \gamma_2 \in \gamma_1$ and $\gamma_1 \cap \gamma_2 \in \gamma_2$ simultaneously would imply

$$\gamma_1 \cap \gamma_2 \in \gamma_1 \cap \gamma_2,$$

a contradiction, we must have exactly one of the three other cases, i.e.

$$\gamma_1 \in \gamma_2 \vee \gamma_1 = \gamma_2 \vee \gamma_2 \in \gamma_1.$$

□

This essentially tells us that when α embeds decidable into an intuitionistic universe, then the initial segment of ordinals before α behaves just like its

⁴⁷Before we prove the linearity of α , we cannot take two arbitrary parameters $x, y \in L_\alpha$ and expect that an instance of Δ_0 -separation with both parameters still hold in L_α , this is because they could be from $\text{def}(L_{\beta_1}), \text{def}(L_{\beta_2})$ respectively, with $\beta_1, \beta_2 \in \alpha$ strongly incomparable, so that we cannot find $\sup\{\beta_1, \beta_2\} \in \alpha$ and the classical argument for Δ_0 -separation will not apply. See Lemma 7.1.5 for more detail.

classical counterpart, even with respect to constructions like Gödel's L . In particular, we immediately have the following analogue of Corollary 5.1.10:

Corollary 7.1.3. *If $\alpha \in \text{Ord}$ embeds decidably into the universe, $f : x \rightarrow \mathcal{P}(\alpha) \cap \text{Ord}$ is pairwise strongly incomparable and $f \in L$, then for any $u \subseteq x$, we have $u \in L$.*

Proof. For any $\beta \in \mathcal{P}(\alpha) \cap \text{Ord}$, we have

$$\beta = \bigcup_{\gamma \in \beta} \gamma^+ = \bigcup_{\gamma \in \beta} \text{def}(L_\gamma) \cap \text{Ord} = L_{\bigcup_{\gamma \in \beta} \gamma^+} \cap \text{Ord} = L_\beta \cap \text{Ord}$$

by Lemma 7.1.2. The claim then follows from Proposition 5.1.9. $\square$

Observe that the relativisation φ^x of a formula φ to any set model x is a Δ_0 -formula, thus $\text{CZF}^- + \text{REM}$ proves that every set model satisfies full classical logic⁴⁸ and every ordinal embeds decidably into the universe.

However, we see that this is not the case for the realisability model:

Proposition 7.1.4. *For any pca (over natural numbers) $\mathcal{A}$, ω embeds decidably into $V(\mathcal{A})$. However, ω^+ does not embed decidably into $V(\text{Kl})$.*

Proof. The first statement follows simply from Theorem 4.3.4. For the second part, take any Π_1 -definable subset $s \subseteq \omega$ in the language of PA that is not recursive, i.e. there is no $e \in \text{Kl}$ such that for any $n \in \omega$, $en \downarrow \in \{0, 1\}$ and $en = 1$ if and only if $n \in s$. For example, simply take the Turing jump of the empty set $\text{TJ}(\emptyset)$ as given explicitly by a universal Π_1 -formula in section 3.4. Now, let φ be the defining formula for s , then we have

$$s^* = \{x \in L_\omega : L_\omega \models x \in \text{Ord} \wedge \varphi^{\text{Ord}}(x)\} \in \text{def}(L_\omega),$$

where φ^{Ord} is the set-theoretic formula obtained from φ by replacing unbounded first-order quantifiers by quantifiers over Ord and arithmetic operations by their Σ_1 -definition as a subset of L_ω .

⁴⁸Pedantically, one might need to distinguish the instances of excluded middle in the axiom scheme for formula φ in the *metatheory* and quantifications over all formulae (or all Gödel numberings) within the set theory itself. However, through the construction of truth predicates (cf. section 4.3 and Lubarsky [24]), one can show that for any (Gödel numbering of) formula φ , the collection of instances $\{\vec{y} \in x^k : \langle x; \in \rangle \models \varphi(\vec{y})\}$ forms a set. Thus, all we need here is the single instance of excluded middle

$$\forall \vec{y} \in x^k (\vec{y} \in \{\vec{y} \in x^k : \langle x; \in \rangle \models \varphi(\vec{y})\} \vee \vec{y} \notin \{\vec{y} \in x^k : \langle x; \in \rangle \models \varphi(\vec{y})\})$$

for an atomic formula in REM.

Now, suppose that ω^+ embeds decidablely into $V(\text{Kl})$, then we must have $V(\text{Kl}) \models \forall n \in \omega (n \in s^* \vee n \notin s^*)$, i.e.

$$V(\text{Kl}) \models \forall n \in \omega (\varphi^\omega(n) \vee \neg \varphi^\omega(n)).$$

Specifically, it is easy to produce canonical names for both ω (i.e. $\bar{\omega}$ as defined in section 4.2) and arithmetic operations on them:

$$\begin{aligned} \oplus &= \{ \langle \mathbf{p}nm, \langle \langle \bar{n}, \bar{m} \rangle^\bullet, \overline{n+m} \rangle^\bullet \rangle : n, m \in \omega \}, \\ \odot &= \{ \langle \mathbf{p}nm, \langle \langle \bar{n}, \bar{m} \rangle^\bullet, \overline{n \cdot m} \rangle^\bullet \rangle : n, m \in \omega \}, \end{aligned}$$

so we can equivalently use explicit names $\bar{\omega}, \oplus, \odot$ in φ^ω , such that it is formally the relativisation of a Π_1 -formula in the language of set theory to the domain ω with additional parameters $\oplus, \odot$.

Now, we know that for any $n, m \in \omega$, $\bar{n} \in \bar{m}$ is realised if and only if $n \in m$, in which case the realiser is always trivially $\mathbf{p}n\iota_r$; similarly, correct instances of $\bar{n} = \bar{m}$, $\langle \langle \bar{n}, \bar{m} \rangle, \bar{k} \rangle \in \oplus$ and $\langle \langle \bar{n}, \bar{m} \rangle, \bar{k} \rangle \in \odot$ are all realised trivially. It follows from easy induction on the complexity of ψ that for any Δ_0 -formula ψ in the language of PA, we have⁴⁹ $d_\psi, k_\psi \in \text{Kl}$ such that for any $n_1, \dots, n_k \in \omega$,

- (i) $d_\psi n_1 \cdots n_k \downarrow \in \{0, 1\}$ and $d_\psi n_1 \cdots n_k = 1$ if and only if $\psi(n_1, \dots, n_k)$ is true on ω in the base theory;
- (ii) if $d_\psi n_1 \cdots n_k = 1$, then $k_\psi n_1 \cdots n_k \downarrow \Vdash \psi^\omega(\bar{n}_1, \dots, \bar{n}_k)$; if $d_\psi n_1 \cdots n_k = 0$, then $\neg \exists e \in \text{Kl } e \Vdash \psi^\omega(\bar{n}_1, \dots, \bar{n}_k)$.

It follows that also for the Π_1 -formula $\varphi(n) = \forall m \psi(n, m)$, $\exists e \in \text{Kl } e \Vdash \varphi^\omega(\bar{n})$ if and only if $\lambda m. k_\psi nm \Vdash \varphi^\omega(\bar{n})$, if and only if $\varphi(n)$ is true on ω in the base theory, if and only if $n \in s$. Thus, if some $d \in \text{Kl}$ satisfies

$$d \Vdash \forall n \in \omega (\varphi^\omega(n) \vee \neg \varphi^\omega(n)),$$

then we must have $\mathbf{p}_0(dn) \downarrow \in \{0, 1\}$ and $\mathbf{p}_0(dn) = 0$ if and only if $n \in s$ for all $n \in \omega$. However, this is impossible as s is not recursive. $\square$

Proposition 7.1.4 additionally shows that being a linear ordinal is not sufficient for the decidable embedding as we defined here. In what follows, we will be concerned with the process of embedding larger ordinals decidablely into a

⁴⁹Observe that this claim is set up in the same style as Proposition 6.2.4, and also proved by a similar induction; the construction of the realisers here, however, will be much simpler. Also cf. Lemma 7.3.10 for a similar process.

realisability model.

Linearity and decidability

We shall point out here that the linearity of an ordinal still simplifies the process of verifying that it embeds decidable into the universe. Examine the following proof of the classical result:

Lemma 7.1.5 (ZF or KP). *For any $\alpha \in \text{Ord}$, $\langle L_\alpha; \in \rangle$ satisfies the axiom scheme of Δ_0 -separation.*

Proof. Let φ be any Δ_0 -formula and $y_1, \dots, y_n \in L_\alpha$, we want to show that

$$x = \{z \in y_1 : \varphi(z, y_1, y_2, \dots, y_n)\} \in L_\alpha.$$

Take $\beta_1, \dots, \beta_n \in \alpha$ such that each $y_i \in \text{def}(L_{\beta_i})$, and let $\beta = \max\{\beta_1, \dots, \beta_n\}$, then we must have $y_1, \dots, y_n \in \text{def}(L_\beta)$, i.e. there will be formulae $\psi_1, \dots, \psi_n$ and parameters $\vec{u}_1, \dots, \vec{u}_n \in L_\beta^{<\omega}$ such that for each y_i ,

$$\forall x \in L_\beta \left(x \in y_i \leftrightarrow \psi_i^{L_\beta}(x, \vec{u}_i) \right).$$

We can now transform φ to an equivalent formula $\tilde{\varphi}$ by the following steps:

- (1) replace every instance of $y_i \in v$ by $\exists z \in v \ z = y_i$, where v denotes any variable;
- (2) replace every instance of $v = y_i$ or $y_i = v$ by $\forall z \in v \ z \in y_i \wedge \forall z \in y_i \ z \in v$;
- (3) replace every instance of $\forall z \in y_i \ \chi(z)$ by $\forall z (z \in y_i \rightarrow \chi(z))$, and every instance of $\exists z \in y_i \ \chi(z)$ by $\exists z (z \in y_i \wedge \chi(z))$;
- (4) replace every instance of $v \in y_i$ by $\psi_i(v, \vec{u}_i)$.

Then $\tilde{\varphi}$ will contain no instances of y_i but the parameters $\vec{u}_i$ instead, and it is easy to verify that

$$x = \left\{ z \in L_\beta : \psi_1^{L_\beta}(z, \vec{u}_1) \wedge \tilde{\varphi}^{L_\beta}(z, \vec{u}_1, \dots, \vec{u}_n) \right\} \in \text{def}(L_\beta) \subseteq L_\alpha. \quad \square$$

As previously mentioned in footnote 47, the proof involves finding a maximal $\beta \in \alpha$ which requires classical logic. However, one can clearly see that if we demand α to be linear beforehand, then IKP suffices for the result. Notably, we can have the following:

Theorem 7.1.6 (IKP). *For any ordinal α , the following are equivalent:*

- (1) α embeds decidable into the universe;
- (2) $L_\alpha \cap \text{Ord}$ is Δ_0 -decidable;
- (3) α is linear and for any $\beta, \gamma \in L_\alpha \cap \text{Ord}$, we have

$$\beta \in \gamma \vee \beta = \gamma \vee \gamma \in \beta,$$

i.e. $L_\alpha \cap \text{Ord}$ is also linear.

Proof. Firstly, it is trivial that (1) implies (2) because $L_\alpha \cap \text{Ord}$ is a Δ_0 -definable subset of L_α . For (2) implying (3), we just need to make the observation that in the proof of Lemma 7.1.2, every instance of excluded middle we use only concerns Δ_0 -formulae on ordinals and thus (2) suffices for the argument.

Finally, assuming (3), take any Δ_0 -formula φ and a corresponding list of sets $y_1, \dots, y_n \in L_\alpha$. We use the linearity of α to find $\beta \in \alpha$ such that $y_1, \dots, y_n \in \text{def}(L_\beta)$, apply the constructions in the classical Lemma 7.1.5 and derive that the set

$$\delta_{\vec{y}} = \{x \in \{\emptyset\} : \varphi(y_1, \dots, y_n)\} \in \text{def}(L_\beta) \subseteq L_\alpha.$$

Note that since $\emptyset \in \text{def}(L_\beta) \subseteq L_\alpha$ trivially, by the linearity assumption we know that $L_\beta \cap \text{Ord} = \emptyset \vee \emptyset \in L_\beta$. The first case is trivial since β is linear, so by Δ_0 -separation from Lemma 7.1.5 we know that any $x \in L_\beta$ implies $\emptyset \in L_\beta \cap \text{Ord}$, thus also $L_\beta = \emptyset$. It follows that $y_1 = \dots = y_n = \emptyset$ and the Δ_0 -formula $\varphi(y_1, \dots, y_n)$ must satisfy excluded middle. Therefore, we can assume the second disjunct $\emptyset \in L_\beta$ without loss of generality and thus $\{\emptyset\} \in \text{def}(L_\beta)$ here.

In this case $\delta_{\vec{y}}$ is well-defined, clearly $\delta_{\vec{y}} \subseteq 1$ and thus $\delta_{\vec{y}} \in L_\alpha \cap \text{Ord}$. Our assumption then implies

$$\delta_{\vec{y}} = \emptyset \vee \emptyset \in \delta_{\vec{y}},$$

which immediately gives $\varphi(y_1, \dots, y_n) \vee \neg\varphi(y_1, \dots, y_n)$ as desired. $\square$

7.2 The uninterpreted set-recursive model

In this section, let $E(X)$ denote the pca of set recursive functions with an additional Σ_1 -class function X , i.e. the class-sized pca given by Σ -inductive definition in section 6.2, except that we omit the clauses for powerset $\bar{\varphi}$.

Here, any set in the base universe V can be a ‘function in the pca $E(X)$ ’, even

though most of them — the ones that do not contain a distinguished operation — will not converge on any input. We can still construct the class of all names, $V(E(X))$, as an inductive definition $I(\Phi)$ where Φ is the Δ_0 -class

$$\{\langle x, y \rangle : x \subseteq V \times y\}.$$

Similar to Corollary 4.2.3, it follows immediately that $x \in V(E(X)) = I(\Phi)$ if and only if $x \subseteq V \times V(E(X))$.

However, since realisers in the pca now range over the entire universe, we observe that $F_{\in}(u, v), G_{=}(u, v)$ as defined in section 4.2 are no longer sets⁵⁰. Thus, they need to be defined through Σ -inductive definitions as well.

We shall let Ψ be the Σ_1 -class of tuples $\langle \langle u, v, d \rangle, y \rangle$ such that $u, v \in V(E(X))$ and there exists $\langle f, x \rangle \in y$ such that

- (i) $\mathbf{p}_0 d \downarrow = f$,
- (ii) for any $\langle g, z \rangle \in u$, $\langle z, x, \mathbf{p}_{10} d g \downarrow \rangle \in y$,
- (iii) for any $\langle g, z \rangle \in x$, $\langle z, u, \mathbf{p}_{11} d g \downarrow \rangle \in y$.

We shall then set $F_{\in} = I(\Psi)$ and

$$G_{=} = \{\langle u, v, d \rangle : u, v \in V(E(X)), \forall \langle f, x \rangle \in u \langle x, v, \mathbf{p}_0 d f \downarrow \rangle \in F_{\in}, \\ \forall \langle f, x \rangle \in v \langle x, u, \mathbf{p}_1 d f \downarrow \rangle \in F_{\in}\}.$$

It follows from Corollary 4.2.2 that $F_{\in}, G_{=} \subseteq V(E(X))^2 \times E(X)$, with also

$$F_{\in} = \{\langle u, v, d \rangle : u, v \in V(E(X)), \exists \langle f, x \rangle \in v (\mathbf{p}_0 d \downarrow = f \wedge \langle u, x, \mathbf{p}_1 d \downarrow \rangle \in G_{=})\}.$$

We define the atomic realisability conditions as

$$\begin{aligned} d \Vdash u \in v & \quad :\Longleftrightarrow \quad \langle u, v, d \rangle \in F_{\in}, \\ d \Vdash u = v & \quad :\Longleftrightarrow \quad \langle u, v, d \rangle \in G_{=}, \end{aligned}$$

and observe that the result of section 4.2 will follow without any change, again except Lemma 4.2.10 for the subset collection scheme. In other words, we have

⁵⁰To be exact, the problem is caused by the definition of $G_{=}$, which expects realisers of $u = v$ to be pairs of functions f, g in the pca. Since the interpretation only cares about values of these functions on f such that $\langle d, x \rangle \in u$ or $\langle d, x \rangle \in v$, if we take some d such that $\neg \exists x \langle d, x \rangle \in u \cup v$ holds, then for any set z , $f' = \lambda a. \mathbf{d}_{\mathbb{N}}(\mathbf{i}_1 1a(\pi dd))1z(fa)$ realises $u = v$ if and only if f does, where $\mathbf{i}_1 1a(\pi dd)$ evaluates to 1 if $a \in \{d\}$, i.e. if $a = d$. It follows that $G_{=}(u, v)$ cannot be a set as long as it is nonempty.

that:

Theorem 7.2.1. *Let φ be any theorem of CZF^- , then CZF^- proves that $V(E(X)) \models \varphi$, i.e. there exists $d \in E(X)$ such that $d \Vdash \varphi$.*

Here $V(E(X))$ still contains incomparable ordinals that we constructed in section 4.4, so it is not a model of REM. However, we can construct a class function of ‘standard names’ from V to $V(E(X))$ recursively as

$$\check{x} = \{\langle y, \check{y} \rangle : y \in x\}.$$

We first observe that:

Lemma 7.2.2. *For any set x, y , if some $d \Vdash \check{x} \in \check{y}$, then $x \in y$; if some $d \Vdash \check{x} = \check{y}$, then $x = y$.*

Proof. We prove the following three statements

$$\begin{aligned} (\exists d d \Vdash \check{x} \in \check{y}) &\rightarrow x \in y, \\ (\exists d d \Vdash \check{y} \in \check{x}) &\rightarrow y \in x, \\ (\exists d d \Vdash \check{x} = \check{y}) &\rightarrow x = y \end{aligned}$$

together by simultaneous set induction on x and y .

Suppose that $d \Vdash \check{x} \in \check{y}$, then there is $z = \mathbf{p}_0 d \downarrow \in y$ such that $\mathbf{p}_1 d \downarrow \Vdash \check{x} = \check{z}$. By the inductive hypothesis, we have $x = z \in y$ as desired. The case where $d \Vdash \check{y} \in \check{x}$ is symmetrical.

Now, suppose that $d \Vdash \check{x} = \check{y}$. For any $z \in x$, we have $\langle z, \check{z} \rangle \in \check{x}$, so $(\mathbf{p}_0 d)z \Vdash \check{z} \in \check{y}$ and by the first clause of inductive hypothesis, we have $z \in y$. Similarly, for any $z \in y$, we have $z \in x$ by the second clause of inductive hypothesis. Thus, $x = y$. $\square$

Corollary 7.2.3. *For any set x , $x \in \text{Ord}$ if and only if $V(E(X)) \models \check{x} \in \text{Ord}$.*

Proof. First notice for any transitive set x , $\lambda yz. \mathbf{p}z \iota_r \Vdash \check{x}$ is transitive. Conversely, if $V(E(X)) \models \check{x}$ is transitive, then for any $y \in x$, $z \in y$, we know that $\check{z} \in \check{x}$ is realised and thus x is transitive by Lemma 7.2.2.

It follows then from the definition of $\check{x}$ that x is a set of transitive sets if and only if $V(E(X))$ realises that $\check{x}$ is a set of transitive sets. $\square$

Lemma 7.2.4 (CZF⁻ + REM). *There is a uniform realiser $d \in E(X)$ such that for every $\alpha \in \text{Ord}$,*

$$d \Vdash \forall \beta, \gamma \in \check{\alpha} (\beta \in \gamma \vee (\beta = \gamma \vee \gamma \in \beta)),$$

i.e. every $\check{\alpha}$ is linear in $V(E(X))$.

Proof. For any $\beta, \gamma \in \alpha$, observe simply from the definition that if $\beta \in \gamma$, then $\mathbf{p}\beta\iota_r \Vdash \check{\beta} \in \check{\gamma}$; if $\beta = \gamma$, then $\iota_r \Vdash \check{\beta} = \check{\gamma}$. Thus, we just need to take $\beta, \gamma \in \alpha$ and compute whether one is an element of the other or that they are equal, and

$$d = \lambda\beta\gamma. \mathbf{d}_{\mathbb{N}}(\mathbf{i}_1 1\beta\gamma)1(\mathbf{p}0(\mathbf{p}\beta\iota_r))(\mathbf{d}_{\mathbb{N}}(\mathbf{i}_1 1\gamma\beta)0(\mathbf{p}1(\mathbf{p}0\iota_r))(\mathbf{p}1(\mathbf{p}1(\mathbf{p}\gamma\iota_r))))$$

realises the desired statement. $\square$

We would like to show that every $\check{\alpha}$ embeds decidablely into $V(E(X))$. For this purpose, we will introduce a new primitive class symbol St in the language of $V(E(X))$, with the following realisability condition:

$$d \Vdash x \in \text{St} \quad :\Longleftrightarrow \quad \exists v (\mathbf{p}_0 d \downarrow = v \wedge \mathbf{p}_1 d \downarrow \Vdash x = \check{v}).$$

Conveniently, we can also introduce the quantifiers

$$\begin{aligned} d \Vdash \forall x \in \text{St } \varphi & \quad :\Longleftrightarrow \quad \forall v dv \downarrow \Vdash \varphi[\check{v}/x], \\ d \Vdash \exists x \in \text{St } \varphi & \quad :\Longleftrightarrow \quad \exists v (\mathbf{p}_0 d \downarrow = v \wedge \mathbf{p}_1 d \downarrow \Vdash \varphi[\check{v}/x]). \end{aligned}$$

and check that they have the same relation to the unbounded quantifiers:

Lemma 7.2.5. *The following are realised in $V(E(X))$:*

$$\begin{aligned} \forall x (x \in \text{St} \rightarrow \varphi(x)) & \leftrightarrow \forall x \in \text{St } \varphi(x), \\ \exists x (x \in \text{St} \wedge \varphi(x)) & \leftrightarrow \exists x \in \text{St } \varphi(x). \end{aligned}$$

Proof. This is completely analogous to Lemma 4.2.5. $\square$

We want $V(E(X))$ to think that St is a transitive model of CZF⁻. For this purpose, we will need that our pca $E(X)$ computes some choice principles. We observe that CZF⁻ is equiconsistent with CZF⁻ + $V = L$ since its L does satisfy all axioms of CZF⁻. Thus we can conveniently move to the base theory CZF⁻ + REM + $V = L$ and observe that CZF⁻ + REM suffices to prove the

classical theorem that there exists a Σ_1 -class bijection between Ord and L (cf. Devlin [10, section II.2]), so that each L_α is mapped to an initial segment of Ord . We shall take $X : V \rightarrow \text{Ord}$ as this bijection.

The transitivity of St is trivial:

Lemma 7.2.6. $V(E(X))$ realises $\forall x \in \text{St} \forall y \in x \ y \in \text{St}$.

Proof. This is simply realised by $\lambda xy. \mathbf{p}y\iota_r$. □

Now, both extensionality and set induction in St are immediate consequences of Lemma 7.2.6. For the pairing axiom:

Lemma 7.2.7. *The pairing axiom in St is realised in $V(E(X))$:*

$$\forall a, b \in \text{St} \exists y \in \text{St} \forall x \in \text{St} (x \in y \leftrightarrow x = a \vee x = b).$$

Proof. For any set a, b , observe that if $y = \{a, b\}$, then

$$\check{y} = \{\langle a, \check{a} \rangle, \langle b, \check{b} \rangle\}.$$

For any x , $\lambda d. \mathbf{p}(\mathbf{d}_\mathbb{N}(\mathbf{p}_0 d)0ab)\iota_r$ realises

$$\check{x} = \check{a} \vee \check{x} = \check{b} \rightarrow \check{x} \in \check{y},$$

since if $\check{x} = \check{a} \vee \check{x} = \check{b}$ is realised, then $x = a$ or b by Lemma 7.2.2; whereas $\lambda d. \mathbf{p}(\mathbf{i}_1 1x(\pi bb))\iota_r$ realises

$$\check{x} \in \check{y} \rightarrow (\check{x} = \check{a} \vee \check{x} = \check{b}),$$

where we check whether x is a or b by testing $x \in \{b\}$. □

For the infinity axiom, it suffices to observe that $V(E(X)) \models \omega \in \text{St}$, which is the case because the name $\bar{\omega}$ we constructed in section 4.2 is precisely $\check{\omega}$. $\text{Ind}(\omega)$ is then a Δ_0 -statement and holds in the transitive class St .

For union and strong collection, we would need the following choice function construction:

Lemma 7.2.8 (CZF⁻ + REM). *When $X : V \rightarrow \text{Ord}$ is a Σ_1 -class bijection, then there is a term $\delta \in E(X)$ such that for any nonempty set s , we have*

$$\delta s \downarrow \in s.$$

Proof. We know that there must exist a single $x \in s$ such that $X(x)$ is the least element inside $\{X(y) : y \in s\}$. We can find this x by first letting $\tau = \lambda s x. \mathbf{d}_{\mathbb{N}}(\boldsymbol{\nu}(\boldsymbol{\rho}(\lambda y. \mathbf{i}_1 1(\mathbf{x}y)(\mathbf{x}x))s))1\emptyset(\boldsymbol{\pi}xx)$, so that

$$\tau s x \downarrow = \begin{cases} \emptyset & \text{if there is } y \in s \text{ such that } X(y) \in X(x), \\ \{x\} & \text{otherwise} \end{cases}$$

for any $x \in s$. Then $\delta = \lambda s. \boldsymbol{\nu}(\boldsymbol{\rho}(\lambda x. \tau s x)s)$ is what we need. $\square$

We then have:

Lemma 7.2.9. *The union axiom in St is realised in $V(E(X))$:*

$$\forall a \in \text{St} \exists b \in \text{St} \forall x \in \text{St} (x \in b \leftrightarrow \exists u \in a \ x \in u).$$

Proof. For any set a , let $b = \bigcup a$. We observe that for any x , $\lambda d. \mathbf{p}x\iota_r$ realises

$$\exists u \in \check{a} \ \check{x} \in u \rightarrow \check{x} \in \check{b}$$

trivially by Lemma 7.2.2. On the other hand, observe that

$$\mathbf{i}_3 a a x = \{u \in a : x \in u\}$$

Using the choice function δ , $\lambda d. \mathbf{p}(\delta(\mathbf{i}_3 a a x))(\mathbf{p}x\iota_r)$ realises

$$\check{x} \in \check{b} \rightarrow \exists u \in \check{a} \ \check{x} \in u. \quad \square$$

Lemma 7.2.10. *The strong collection scheme in St is realised in $V(E(X))$: for any $a \in \text{St}$,*

$$\begin{aligned} & \forall x \in a \exists y \in \text{St} \theta^{\text{St}}(x, y) \\ & \rightarrow \exists b \in \text{St} (\forall x \in a \exists y \in b \theta^{\text{St}}(x, y) \wedge \forall y \in b \exists x \in a \theta^{\text{St}}(x, y)). \end{aligned}$$

Proof. For any set a , suppose that $f \Vdash \forall x \in \check{a} \exists y \in \text{St} \theta^{\text{St}}(x, y)$, and let $b = \boldsymbol{\rho}(\lambda x. \mathbf{p}_0(fx))a$. It is easy to check that

$$f \Vdash \forall x \in \check{a} \exists y \in \check{b} \theta^{\text{St}}(x, y).$$

On the other hand, let $g = \lambda a y. \boldsymbol{\nu}(\boldsymbol{\rho}(\lambda x. \mathbf{i}_1(\boldsymbol{\pi}xx)(\mathbf{p}_0(fx))(\boldsymbol{\pi}yy))a)$, so that

$$g a y \downarrow = \{x \in a : \mathbf{p}_0(fx) = y\},$$

then $\lambda y. \mathbf{p}(\delta(gay))(\mathbf{p}_1(f(\delta(gay))))$ realises

$$\forall y \in \check{b} \exists x \in \check{a} \theta^{\text{St}}(x, y). \quad \square$$

Finally, the binary intersection axiom is trivial:

Lemma 7.2.11. *The binary intersection axiom in St is realised in $V(E(X))$:*

$$\forall a, b \in \text{St} \exists y \in \text{St} \forall x \in \text{St} (x \in y \leftrightarrow x \in a \wedge x \in b).$$

Proof. For any set a, b , take $y = a \cap b$. For any x , simply by Lemma 7.2.2, $\lambda d. \mathbf{p}(\mathbf{p}x\iota_r)(\mathbf{p}x\iota_r)$ realises

$$\check{x} \in \check{y} \rightarrow \check{x} \in \check{a} \wedge \check{x} \in \check{b},$$

whereas $\lambda d. \mathbf{p}x\iota_r$ realises

$$\check{x} \in \check{a} \wedge \check{x} \in \check{b} \rightarrow \check{x} \in \check{y}. \quad \square$$

Therefore, we have verified that:

Theorem 7.2.12. *Let φ be any theorem of CZF^- , then $\text{CZF}^- + \text{REM} + V = L$ proves $V(E(X)) \models \varphi^{\text{St}}$, where $X : V \rightarrow \text{Ord}$ is a Σ_1 -class bijection.*

Corollary 7.2.13 ($\text{CZF}^- + \text{REM} + V = L$). *For any $\alpha \in \text{Ord}$, $\check{\alpha}$ embeds decidablely into $V(E(X))$.*

Proof. For every $\alpha \in \text{Ord}$, $V(E(X))$ realises that $\check{\alpha}$ is an ordinal by Corollary 7.2.3, and realises that $\check{\alpha}$ is linear by Lemma 7.2.4. Thus by Theorem 7.1.6, it suffices to work inside $V(E(X))$ and show that $L_{\check{\alpha}} \cap \text{Ord}$ is also linear.

Now, we use the fact that $V(E(X)) \models (\text{CZF}^-)^{\text{St}}$, thus there is a well-defined set $L_{\check{\alpha}}^{\text{St}}$ in the class St . Since Gödel's L is a Σ_1 -definable hierarchy and thus upwards absolute across transitive classes, it follows that $L_{\check{\alpha}} = L_{\check{\alpha}}^{\text{St}}$. Then

$$L_{\check{\alpha}} \cap \text{Ord} = L_{\check{\alpha}}^{\text{St}} \cap \text{Ord} \in \text{St}$$

is an ordinal in St , and by Corollary 7.2.3, $L_{\check{\alpha}} \cap \text{Ord} = \check{\beta}$ for some $\beta \in \text{Ord}$ in the base theory. Therefore, by Lemma 7.2.4 again we know that $L_{\check{\alpha}} \cap \text{Ord}$ is linear. $\square$

However, the system $V(E(X))$ also has its problems. Firstly, while we have

Corollary 7.1.3 for arbitrarily large ordinals $\check{\alpha}$, one still cannot construct incomparable coding arguments in the form of section 5.4 here. This is because every set-theoretic function, when supplied as a parameter, is computable in the pca $E(X)$. Consequently:

Lemma 7.2.14 (CZF⁻ + REM + $V = L$). *For any set x and name $y \in V(E(X))$, suppose that for any $z \in x$ there exists f such that $\langle f, \check{z} \rangle \in y$, then $V(E(X)) \models \check{x} \subseteq y$.*

It follows that if the name y can be seen as a surjective set-theoretic function onto $\{\check{z} : z \in x\}$, then $V(E(X)) \models \check{x} = y$.

Proof. By REM and $V = L$ together, we have a global axiom of choice and can construct a set-theoretic function h on domain x such that for every $z \in x$, $\langle h(z), \check{z} \rangle \in y$. Since $E(X)$ can evaluate set-theoretic functions (cf. footnote 44 on page 142), for every $z \in x$ it suffices to see that $\mathbf{p}g\iota_r \Vdash \check{z} \in y$ where $g = h(z)$. Then $V(E(X)) \models \check{x} \subseteq y$.

Conversely, if y is a surjective function onto $\{\check{z} : z \in x\}$, then we can easily build

$$y' = \{\langle f, z \rangle : \langle f, \check{z} \rangle \in y\},$$

by observing that z can be obtained back as $z = \{y : \langle y, \check{y} \rangle \in \check{z}\}$. For every $\langle f, \check{z} \rangle \in y$, then $\mathbf{p}z\iota_r \Vdash \check{z} \in \check{x}$, where $z = y'(f)$ can be computed from the index f . It follows that $V(E(X)) \models y \subseteq \check{x}$ and thus $\check{x} = y$. $\square$

By Lemma 7.2.14, imitating the constructions of names in $\mathcal{P}(\omega) \cap \text{Ord}$ in section 5.4 would only lead to names in $V(E(X))$ that are (realised to be) equal to $\check{\omega}$. Thus we cannot construct ordinal names $\beta \in V(E(X))$ that are realised as incomparable, proper subsets of $\check{\alpha}$ easily as we did in section 5.4.

Secondly, we also point out $V(E(X))$ is one construction that cannot accommodate the powerset axiom — even when the base theory does admit the powerset axiom — and this limits its adaptability to the extensions of CZF⁻. Specifically, we can prove the following:

Theorem 7.2.15 (CZF⁻). *$V(E(X))$ realises the negation of the powerset axiom. Namely, it realises that*

$$\neg \exists y \forall x (x \in y \leftrightarrow x \subseteq 1).$$

Proof. It suffices to derive a contradiction from the assumption that there is

$y \in V(E(X))$ with some $f \in E(X)$ realising

$$x \in y \leftrightarrow x \subseteq \bar{1}$$

for all $x \in V(E(X))$, where $\bar{1} = \{\langle \emptyset, \emptyset \rangle\}$ is as given in section 4.2.

Let $g = \lambda e. \mathbf{p}\emptyset\iota_r$ such that for any set a ,

$$g \Vdash \{\langle a, \emptyset \rangle\} \subseteq \bar{1}.$$

It follows that there must exist some $\langle d, z \rangle \in y$ such that

$$\mathbf{p}_{11}(\mathbf{p}_1fg) \Vdash z \subseteq \{\langle a, \emptyset \rangle\},$$

where z is nonempty because we also realise $\{\langle a, \emptyset \rangle\} \subseteq z$. We now consider

$$s = \{\mathbf{p}_0(\mathbf{p}_{11}(\mathbf{p}_1fg)e) : \langle d, z \rangle \in y, \langle e, u \rangle \in z\},$$

where we know that the computation must converge and thus s exists by strong collection. Then also

$$\mathbf{p}_{11}(\mathbf{p}_1fg) \Vdash z \subseteq \{\langle s, \emptyset \rangle\},$$

for some $\langle d, z \rangle \in y$ where z is nonempty, i.e. for every $\langle e, u \rangle \in z$, we have $\mathbf{p}_0(\mathbf{p}_{11}(\mathbf{p}_1fg)e) \downarrow = s$. However, by our definition of s above, we also have

$$\mathbf{p}_0(\mathbf{p}_{11}(\mathbf{p}_1fg)e) \in s,$$

a contradiction. □

7.3 Realisability over α -recursion

We will suggest that the problems of $V(E(X))$ can be resolved by considering instead a large set-sized pca, for example the (weak) pca of α -recursive functions for an admissible ordinal α .

Following Sacks [40, Chapter VII], we say that an ordinal α is Σ_1 -*admissible* (or just *admissible*) if L_α satisfies the Δ_0 -collection axiom. Working in the base theory $\text{CZF}^- + \text{REM}$, we observe that REM suffices to ensure that $\langle L_\alpha; \in \rangle$ satisfies full excluded middle, whereas Lemma 7.1.5 applies to justify that L_α also satisfies Δ_0 -separation. Thus, it is easy to see that any $\alpha \in \text{Ord}$ is Σ_1 -admissible if and only if L_α is a model of classical KP – Infinity.

An α -recursive function is essentially a partial function on L_α that is Σ_1 -definable in the structure $\langle L_\alpha; \in \rangle$. Formally, we want to fix a universal Σ_1 -formula $\pi(e, m, a, b)$ in the language of set theory, such that for any Σ_1 -formula $\varphi(m, a, b)$ with all free variables indicated, KP (even without the axiom of infinity) proves that there exists a set e such that

$$\forall m \forall a \forall b (\varphi(m, a, b) \leftrightarrow \pi(e, m, a, b)).$$

One could follow the construction in [4, section V.1] or simply construct a truth predicate for Δ_0 -formulae by Σ -recursion, using an approach similar to the predicate $\text{Tr}_{\varphi\top}$ in section 4.3.

We then perform the same stipulations as we did in section 3.2 for Σ_1^1 -definable functions in second-order arithmetic. We let ΣL_α denote the domain L_α with a partial application function given as $ea \downarrow = b$ if and only if $b \in L_\alpha$ is the *unique* element such that

$$\pi^{L_\alpha}(e_0, e_1, a, b)$$

holds, where $e = \langle e_0, e_1 \rangle$ is the set-theoretic, Kuratowski ordered pair. We observe that:

Lemma 7.3.1. *Given any Σ_1 -formula $\varphi(m, a, b)$ with all free variables indicated, then there exists $e \in L_\alpha$ such that for any $m, a \in L_\alpha$, $em \downarrow$ always holds, and $ema \downarrow = b$ if and only if $b \in L_\alpha$ is the unique element such that $\varphi^{L_\alpha}(m, a, b)$ holds.*

Proof. Just like Lemma 3.2.1, pick $f, n \in L_\alpha$ by universality such that

$$\begin{aligned} \forall m \forall a \forall b (b = \langle m, a \rangle &\leftrightarrow \pi(f, m, a, b)), \\ \forall m \forall a \forall b (\varphi(m, a, b) &\leftrightarrow \pi(n, m, a, b)), \end{aligned}$$

where the set-theoretic ordered pair is expressed using its Δ_0 defining formula. Let $e = \langle f, n \rangle$, then $em \downarrow = \langle n, m \rangle$ for any $m \in L_\alpha$ and thus we have $ema \downarrow = b$ precisely when $b \in L_\alpha$ is the unique element such that $\varphi^{L_\alpha}(m, a, b)$ holds. $\square$

Corollary 7.3.2. *For any admissible ordinal α , ΣL_α is a weak pca over natural numbers.*

Proof. First observe that any $\alpha \in \omega$ is not admissible. Thus by REM, we know that $\omega \subseteq \alpha$ and hence $\omega \subseteq L_\alpha$ as well.

We now use Lemma 7.3.1 to identify the distinguished elements in the algebra. For operations that involve one parameter only, we simply ignore the parameter m in the proposition:

- (i) let $s \in L_\alpha$ be the index such that for any $m, a \in L_\alpha$, $sma \downarrow = b$ if and only if $b = a^+ = a \cup \{a\}$, and set $\mathbf{s}_\mathbb{N} = s\emptyset$;
- (ii) let $p \in L_\alpha$ be the index such that for any $m, a \in L_\alpha$, $pma \downarrow = b$ if and only if $a = b^+$, and set $\mathbf{p}_\mathbb{N} = p\emptyset$;

For $\mathbf{k}$ which involves two parameters, we use precisely the parameters in Lemma 7.3.1:

- (iii) let $\mathbf{k} \in L_\alpha$ simply be the index such that for any $a, b \in L_\alpha$, $\mathbf{k}ab \downarrow = c$ if and only if $c = a$.

For $\mathbf{s}$ which involves three parameters, we invoke the proposition twice:

- (iv) let $s' \in L_\alpha$ be the index such that for any $a, b \in L_\alpha$, $s'ab \downarrow = c$ if and only if $c \in L_\alpha$ is the unique element such that there exist $d, e \in L_\alpha$ and it is possible to decompose

$$a = \langle \langle p_0, p_1 \rangle, \langle q_0, q_1 \rangle \rangle, \quad d = \langle d_0, d_1 \rangle,$$

such that the following holds⁵¹:

$$\pi^{L_\alpha}(a_{00}, a_{01}, b, d) \wedge \pi^{L_\alpha}(a_{10}, a_{11}, b, e) \wedge \pi^{L_\alpha}(d_0, d_1, e, c).$$

In other words, if $\langle p_0, p_1 \rangle b \downarrow = d$, $\langle q_0, q_1 \rangle b \downarrow = e$ and $de \downarrow = c$, then also $s'ab \downarrow = c$. Now just let $\mathbf{s} \in L_\alpha$ be such that $\mathbf{s}ab \downarrow = s' \langle a, b \rangle$.

Finally, similarly for $\mathbf{d}_\mathbb{N}$ which involves four parameters:

- (v) let $d' \in L_\alpha$ be the index such that for any $a, b \in L_\alpha$, if we can decompose $a = \langle \langle p, q \rangle, r \rangle$, then

$$d'ab \downarrow = \begin{cases} r & \text{if } p = q, \\ b & \text{if } p \neq q. \end{cases}$$

Then let $d'' \in L_\alpha$ be such that $d''ab \downarrow = d' \langle a, b \rangle$ and finally $\mathbf{d}_\mathbb{N} \in L_\alpha$ be such that $\mathbf{d}_\mathbb{N}ab \downarrow = d'' \langle a, b \rangle$. $\square$

We will also stipulate that the pairing function $\mathbf{p} \in \Sigma L_\alpha$ computes precisely

⁵¹Just as in section 3.2, the existence of a unique c here does not also guarantee the uniqueness of the intermediate values d and e . Thus the existence of a unique c does not imply $a_0b(a_1b) \downarrow$ and thus ΣL_α as constructed here is only a weak pca.

the Kuratowski ordered pair, i.e. $\mathbf{p}ab \downarrow = \langle a, b \rangle$.

We can work with precisely the realisability model $V(\Sigma L_\alpha)$ as constructed in section 4.2. Just as in section 7.2, we define standard names $\check{x}$ for any $x \in L_\alpha$ recursively as

$$\check{x} = \{ \langle y, \check{y} \rangle : y \in x \}.$$

But now we have sets of ‘all standard names’ or ‘all standard ordinals’ as well, given as

$$\begin{aligned} \check{L}_\alpha &= \{ \langle x, \check{x} \rangle : x \in L_\alpha \} \in V(\Sigma L_\alpha), \\ \check{\alpha} &= \{ \langle \beta, \check{\beta} \rangle : \beta \in \alpha \} \in V(\Sigma L_\alpha). \end{aligned}$$

When $\alpha > \omega$ is admissible, i.e. L_α is a model of KP, we want to repeat the process in section 7.2 and check that $\check{\alpha}$ embeds decidablely into $V(\Sigma L_\alpha)$. We shall first note that KP suffices to define all the set-recursive functions, i.e. $E(\emptyset)$, as a Σ_1 -class. It follows from Lemma 7.3.1 that every specific function computable by $E(\emptyset)$ inside L_α is also computable by some α -recursive $e \in \Sigma L_\alpha$. Thus we can have most results from section 7.2 by essentially the same proof. We have that:

Lemma 7.3.3. *For any $x, y \in L_\alpha$, if some $d \Vdash \check{x} \in \check{y}$, then $x \in y$; if some $d \Vdash \check{x} = \check{y}$, then $x = y$.*

Corollary 7.3.4. *For any $x \in L_\alpha$, $x \in \text{Ord}$ if and only if $V(\Sigma L_\alpha) \models \check{x} \in \text{Ord}$.*

Lemma 7.3.5. *$V(\Sigma L_\alpha)$ realises that $\check{\alpha}$ is linear.*

Proof. Just take the construction from Lemma 7.2.4. Note the uniformity in Lemma 7.2.4 corresponds precisely to the fact here that the realiser does not need to take α itself as a parameter (which it cannot, since $\alpha \notin L_\alpha$). $\square$

Now, the analogue of St here is simply the set $\check{L}_\alpha$, and $V(\Sigma L_\alpha)$ should realise that $\langle \check{L}_\alpha; \in \rangle$ is a model of CZF^- . This would again require a choice principle, but we should observe that the following is provable in $\text{CZF}^- + \text{REM}$:

Proposition 7.3.6 ([40, Theorem VII.1.8]). *When α is any admissible ordinal, there exists an α -recursive bijection from α onto L_α .*

Proof sketch. Using the fact that L_α is a model of KP – Infinity, then the claim is essentially the existence of a Σ_1 -class bijection between Ord and L in KP – Infinity, which we already discussed in section 7.2. $\square$

Corollary 7.3.7. *There is a term $\delta \in \Sigma L_\alpha$ such that for any nonempty $s \in L_\alpha$, we have*

$$\delta s \downarrow \in s.$$

Proof. This is completely analogous to Lemma 7.2.8. $\square$

We can then verify that all axioms of CZF^- hold for $\check{L}_\alpha$ in the same way as section 7.2, where we should specifically observe that the proof for the strong collection scheme in St (correspondingly $\check{L}_\alpha$ here), Lemma 7.2.10, does not require⁵² collection from the base theory, even though the fact that the entire uninterpreted model ($V(E(X))$ or $V(\Sigma L_\alpha)$) satisfies a collection scheme would involve strong collection in the base theory. Therefore, the constructions in section 7.2 can be replicated under just the assumption that L_α satisfies KP and we have:

Theorem 7.3.8. *For admissible ordinal $\alpha > \omega$, $V(\Sigma L_\alpha)$ sees that $\check{L}_\alpha$ is a transitive set; and let φ be any theorem of CZF^- , then $\text{CZF}^- + \text{REM}$ proves $V(\Sigma L_\alpha) \models \varphi^{\check{L}_\alpha}$.*

Corollary 7.3.9 ($\text{CZF}^- + \text{REM}$). *For any admissible ordinal α , $\check{\alpha}$ embeds decidable into $V(\Sigma L_\alpha)$.*

Proof. The case $\alpha = \omega$ is already covered by Proposition 7.1.4. For $\alpha > \omega$, $V(\Sigma L_\alpha)$ realises that $\check{\alpha}$ is linear by Lemma 7.3.5. By Theorem 7.1.6, it suffices to work inside $V(\Sigma L_\alpha)$ and show that $L_{\check{\alpha}} \cap \text{Ord}$ is also linear.

We take any ordinal $\gamma \in L_{\check{\alpha}}$, then there exists $\beta \in \check{\alpha}$ such that $\gamma \in \text{def}(L_\beta)$. However, using the fact that $V(\Sigma L_\alpha) \models (\text{CZF}^-)^{\check{L}_\alpha}$ (and that trivially $\check{\alpha} \subseteq \check{L}_\alpha$), we know there exists some well-defined $L_{\beta^+}^{\check{L}_\alpha} \in \check{L}_\alpha$, and

$$\text{def}(L_\beta) = L_{\beta^+} = L_{\beta^+}^{\check{L}_\alpha}$$

by upward absoluteness. It follows that $\gamma \in \check{L}_\alpha$ and thus $V(\Sigma L_\alpha)$ realises $\gamma = \check{\delta}$ for some $\delta \in \alpha$ in the base theory, by Corollary 7.3.4. In other words, $V(\Sigma L_\alpha)$ realises that $L_{\check{\alpha}} \cap \text{Ord} \subseteq \check{\alpha}$ and its linearity follows immediately. $\square$

⁵²To be more precise, the fact that we can compute

$$\rho xy = \{xu : u \in y\}$$

invokes an instance of Σ_1 -collection, which is provided in any model of IKP or KP, such as L_α here.

Incomparable ordinals in $V(\Sigma L_\alpha)$

Fix some admissible ordinal $\alpha > \omega$. We are now ready to build a pairwise strongly incomparable function f to apply Corollary 7.1.3. Just like in section 5.4, we need to build some non- α -recursive objects. First observe that:

Lemma 7.3.10. *$V(\Sigma L_\alpha)$ realises $\check{L}_\alpha = L_{\check{\alpha}}$. Additionally, for any Δ_0 -formula $\varphi(x_1, \dots, x_n)$ with all free variables indicated, there exist terms $d_\varphi, k_\varphi \in \Sigma L_\alpha$ such that for any $a_1, \dots, a_n \in L_\alpha$,*

- (i) $d_\varphi a_1 \cdots a_n \downarrow \in \{0, 1\}$ and $d_\varphi a_1 \cdots a_n = 1$ if and only if $\varphi(a_1, \dots, a_n)$ holds;
- (ii) if $d_\varphi a_1 \cdots a_n = 1$, then $k_\varphi a_1 \cdots a_n \downarrow \Vdash \varphi(\check{a}_1, \dots, \check{a}_n)$; if $d_\varphi a_1 \cdots a_n = 0$, then $\neg \exists e e \Vdash \varphi(\check{a}_1, \dots, \check{a}_n)$.

It follows immediately that $\varphi(a_1, \dots, a_n)$ if and only if $\exists e e \Vdash \varphi(\check{a}_1, \dots, \check{a}_n)$.

Proof. The existence of d_φ, k_φ is proved in a similar style to Proposition 6.2.4, although the case here is much simpler since we are not dealing with weak realisability. The atomic case is simply due to Lemma 7.3.3 and the trivial realisers of $\check{x} \in \check{y}$ and $\check{x} = \check{y}$. In the binary connective cases, it suffices for d_φ to compute binary operations since L_α is a model of classical logic. The bounded quantifier cases are due to the fact that we can search through a set $a \in L_\alpha$ (which corresponds to the ‘finite sets’ in usual Turing machine computations) in ΣL_α using the operation $\boldsymbol{\rho}$ (cf. the corresponding constructions in Proposition 6.2.4).

Now, it suffices to prove by set induction on β that for any $\beta \in \alpha^+$, $V(\Sigma L_\alpha) \models \check{L}_\beta = L_{\check{\beta}}$ where

$$\check{L}_\beta = \{\langle x, \check{x} \rangle : x \in L_\beta\}.$$

The only non-trivial case is the successor case, where we need to verify that if $y = \text{def}(x)$ where $x \in L_\alpha$, then $V(\Sigma L_\alpha) \models \check{y} = \text{def}(\check{x})$. This is precisely because, for any $z \in y = \text{def}(x)$, we have that

$$z = \{u \in x : \varphi^x(u, v_1, \dots, v_n)\}$$

for $v_1, \dots, v_n \in x$. Now, the relativisation of φ (to a set variable) will be a Δ_0 -formula, thus

$$\lambda u. \mathbf{p}(\lambda d. k_{\varphi^x} x u v_1 \cdots v_n)(\lambda d. \mathbf{p} u \iota_r)$$

will realise $\forall u \in \check{x} (u \in \check{z} \leftrightarrow \varphi^{\check{x}}(u, \check{v}_1, \dots, \check{v}_n))$, and thus $V(\Sigma L_\alpha) \models \check{z} \in \text{def}(\check{x})$.

Conversely, suppose that $v \in V(\Sigma L_\alpha)$ is a name such that⁵³

$$V(\Sigma L_\alpha) \models v = \{u \in \check{x} : \varphi^{\check{x}}(u, \check{v}_1, \dots, \check{v}_n)\},$$

for some $v_1, \dots, v_n \in x$. We use Theorem 7.3.8 that $\check{L}_\alpha$ is a model of CZF^- to derive that, given $\check{x} \in \check{L}_\alpha$, it must follow that $v \in \text{def}(\check{x}) \in \check{L}_\alpha$, i.e. there is $z \in x$ such that $v = \check{z}$, as realised in $V(\Sigma L_\alpha)$. It follows again from φ^x being Δ_0 that we will have

$$\forall u \in x (u \in z \leftrightarrow \varphi^x(u, v_1, \dots, v_n)),$$

i.e. $z \in y = \text{def}(x)$ and thus $v = \check{z} \in \check{y}$ is realised. $\square$

Corollary 7.3.11. *Consider a relation $R \subseteq L_\alpha^n$ that is Σ_1 -definable in L_α . Then $V(\Sigma L_\alpha)$ realises that the corresponding name*

$$\overline{R} = \{\langle \langle x_1, \dots, x_n \rangle, \langle \check{x}_1, \dots, \check{x}_n \rangle^\bullet \rangle : \langle x_1, \dots, x_n \rangle \in R\} \in V(\Sigma L_\alpha)$$

lies in L .

Proof. Let R be defined by

$$\langle x_1, \dots, x_n \rangle \in R \leftrightarrow \exists y \in L_\alpha \varphi(x_1, \dots, x_n, y),$$

where φ is Δ_0 . Let $h : \alpha \rightarrow L_\alpha$ be the α -recursive bijection from α onto L_α given by Proposition 7.3.6, then for any $\langle x_1, \dots, x_n \rangle \in R$, there exists a unique $y \in L_\alpha$ such that $\varphi(x_1, \dots, x_n, y)$ holds and y is h -least, i.e. for any $\beta \in h^{-1}(y)$, we have $\neg \varphi(x_1, \dots, x_n, h(\beta))$. Observe that here the unique y is defined by a Σ_1 -formula relativised to L_α , thus Lemma 7.3.1 applies and there exists $g \in \Sigma L_\alpha$ such that for any $\langle x_1, \dots, x_n \rangle \in R$,

$$\varphi(x_1, \dots, x_n, gx_1 \cdots x_n \downarrow).$$

By Lemma 7.3.10, then

$$\mathbf{p}(gx_1 \cdots x_n)(k_\varphi x_1 \cdots x_n(gx_1 \cdots x_n)) \Vdash \exists y \in \check{L}_\alpha \varphi(\check{x}_1, \dots, \check{x}_n, y).$$

It is easy to verify that conversely, for any $x_1, \dots, x_n \in L_\alpha$, if $V(\Sigma L_\alpha)$ real-

⁵³Formally, the formula φ here is also represented by its Gödel numbering as a name in $V(\Sigma L_\alpha)$. However, since $V(\Sigma L_\alpha)$ has the same ω as the base theory, it is not hard to ensure that the numbering $\check{n} \in \check{\omega}$ for φ in $V(\Sigma L_\alpha)$ and the corresponding n in the base theory will represent precisely the same formula.

ises $\exists y \in \check{L}_\alpha \varphi(\check{x}_1, \dots, \check{x}_n, y)$, then $\exists y \in L_\alpha \varphi(x_1, \dots, x_n, y)$ is true in the base theory by Lemma 7.3.10 again. Thus $V(\Sigma L_\alpha)$ realises

$$\forall x_1, \dots, x_n \in \check{L}_\alpha \left(\langle x_1, \dots, x_n \rangle \in \bar{R} \leftrightarrow \exists y \in \check{L}_\alpha \varphi(\check{x}_1, \dots, \check{x}_n, y) \right),$$

i.e. $\bar{R} \in \text{def}(\check{L}_\alpha)$. By Lemma 7.3.10, $\check{L}_\alpha = L_{\check{\alpha}}$ and thus $\bar{R} \in \text{def}(L_{\check{\alpha}}) \subseteq L$. $\square$

We can now proceed with the constructions. To begin, we work inside L_α as a model of KP, fix a Σ_1 -class bijection H from $\text{Ord}^{L_\alpha} = \alpha$ onto $V^{L_\alpha} = L_\alpha$ as mentioned in Proposition 7.3.6, and construct some $g_\beta \in L_\alpha$ for each $\beta \in \alpha$ through Σ -recursion as follows:

For each β , first let $g_\beta^* = \bigcup_{\gamma \in \beta} g_\gamma$ and $\beta^* = \{\gamma \in \alpha : \exists a \in L_\alpha \langle \gamma, a \rangle \in g_\beta^*\}$ be the domain of g_β^* . Consider the following cases:

(i) If $H(\beta) = \langle e, \langle \gamma, \delta \rangle \rangle$ for $e \in L_\alpha$ and $\gamma \neq \delta \in \alpha$, then let

$$\eta = \bigcup \{ \varepsilon^+ : \exists \zeta \in \alpha (\langle \zeta, \langle \gamma, \varepsilon \rangle \rangle \in g_\beta^* \vee \langle \zeta, \langle \delta, \varepsilon \rangle \rangle \in g_\beta^*) \}.$$

We must have $\eta \in \alpha$ since $g_\beta^* \in L_\alpha$ and L_α is a model of KP. Take the unique enumeration (i.e. an order-preserving bijection)

$$j_\beta : \mu_\beta \rightarrow \{ \varepsilon \in \eta : \neg \exists \zeta \in \alpha \langle \zeta, \langle \gamma, \varepsilon \rangle \rangle \in g_\beta^* \}$$

in the usual ordering, where $\mu_\beta \in \alpha$ is a proper initial segment, and let

$$\begin{aligned} g_\beta &= g_\beta^* \cup \{ \langle \beta^* + \varepsilon, \langle \gamma, j_\beta(\varepsilon) \rangle \rangle : \varepsilon \in \mu_\beta \} \\ &\quad \cup \{ \langle \beta^* + \mu_\beta, \langle \gamma, \eta \rangle \rangle, \langle (\beta^* + \mu_\beta)^+, \langle \delta, \eta \rangle \rangle \}. \end{aligned}$$

(ii) If $H(\beta) = \langle \gamma, \delta \rangle$ for $\gamma, \delta \in \alpha$, and there does not exist $\varepsilon \in \alpha$ such that $\langle \varepsilon, \langle \gamma, \delta \rangle \rangle \in g_\beta^*$, then let

$$g_\beta = g_\beta^* \cup \{ \langle \beta^*, \langle \gamma, \delta \rangle \rangle \}.$$

(iii) Otherwise, simply let $g_\beta = g_\beta^*$.

Observe that we can distinguish cases (i) and (ii) because in classical logic, any ordinal γ must either be empty or contain the empty set $\emptyset$, whereas Kuratowski ordered pairs are always nonempty sets that do not contain $\emptyset$ as an element. Thus it cannot be the case that some $\langle e, \langle \gamma, \delta \rangle \rangle = \langle \gamma', \delta' \rangle$.

We first check that:

Lemma 7.3.12 (CZF⁻ + REM). *The Σ_1 -definable $g = \bigcup_{\beta \in \alpha} g_\beta$ is a bijection between α and $\alpha \times \alpha$.*

Proof. It is easy to check by set induction that each g_β^* and g_β are functions on proper initial segments of α , with $\gamma \in \beta \in \alpha$ implying $g_\gamma \subseteq g_\beta$. Thus, g must also be a function on an initial segment of α .

Now, clause (ii) in the definition of g_β ensures that g is a surjection onto $\alpha \times \alpha$. However, g is a Σ_1 -class in the set model L_α of KP, and by Δ_0 -collection, there cannot be a surjection from an ordinal (in α) onto $(\text{Ord} \times \text{Ord})^{L_\alpha}$. Thus we know that the domain of g must be α itself.

Finally, for the injectivity of g , by the linearity of ordinals from REM, it suffices to check that every g_β is injective. In clause (i), this is ensured by the choice of η , since for any $\langle \zeta, \langle \gamma, \varepsilon \rangle \rangle$ or $\langle \zeta, \langle \delta, \varepsilon \rangle \rangle \in g_\beta^*$, we must have $\varepsilon \in \eta$ and thus $\langle \gamma, \varepsilon \rangle \neq \langle \gamma, \eta \rangle$, $\langle \delta, \varepsilon \rangle \neq \langle \delta, \eta \rangle$. In clause (ii), this is ensured simply by stipulation. $\square$

By Lemma 7.3.1, both this function g and its inverse are then also computed by some terms in ΣL_α . We can denote $G, G^{-1} \in \Sigma L_\alpha$ such that

$$\begin{aligned} G\beta \downarrow &= g(\beta), \\ G^{-1}\beta\gamma \downarrow &= g^{-1}(\beta, \gamma) \end{aligned}$$

for any $\beta, \gamma \in \alpha$. On the other hand, by Corollary 7.3.11, there will also be a name $\overline{G} \in V(\Sigma L_\alpha)$ that performs the same computations on $\check{\alpha}$, such that $V(\Sigma L_\alpha) \models \overline{G} \in L$.

Next, in the base theory CZF⁻ + REM, outside of L_α , we will recursively construct a corresponding sequence of objects h_β — which should be functions on the same domain as g_β — for each $\beta \in \alpha$ as follows:

For each β , first let $h_\beta^* = \bigcup_{\gamma \in \beta} h_\gamma$. Consider the following cases:

- (i) If $H(\beta) = \langle e, \langle \gamma, \delta \rangle \rangle$ for $e \in L_\alpha$ and $\gamma \neq \delta \in \alpha$, we take j_β, μ_β from the corresponding clause in the construction of g_β and let

$$h'_\beta = h_\beta^* \cup \{ \langle \beta^* + \varepsilon, \emptyset \rangle : \varepsilon \in \mu_\beta \}.$$

Evaluate eh'_β in ΣL_α . If $eh'_\beta \downarrow = h'_\beta \cup \{ \langle \beta^* + \mu_\beta, f \rangle \}$ for some $f \in L_\alpha$, then let

$$h_\beta = h'_\beta \cup \{ \langle \beta^* + \mu_\beta, f^+ \rangle, \langle (\beta^* + \mu_\beta)^+, \emptyset \rangle \},$$

where $f^+ = f \cup \{f\}$. If it does not converge or if its output does not have the correct form — which we can test by REM — simply let

$$h_\beta = h'_\beta \cup \{\langle \beta^* + \mu_\beta, \emptyset \rangle, \langle (\beta^* + \mu_\beta)^+, \emptyset \rangle\}.$$

(ii) If $H(\beta) = \langle \gamma, \delta \rangle$ for $\gamma, \delta \in \alpha$, and there does not exist $\varepsilon \in \alpha$ such that $\langle \varepsilon, \langle \gamma, \delta \rangle \rangle \in h_\beta^*$, let $h_\beta = h_\beta^* \cup \{\langle \beta^*, \emptyset \rangle\}$.

(iii) Otherwise, simply let $h_\beta = h_\beta^*$.

Lemma 7.3.13. *The function $h = \bigcup_{\beta \in \alpha} h_\beta$ is a function from α to L_α . Additionally, for any $e \in \Sigma L_\alpha$, and $\gamma, \delta \in \alpha$ where $\gamma \neq \delta$, there exists $\eta \in \alpha$ such that*

$$e(h \upharpoonright G^{-1}\gamma\eta) \downarrow \rightarrow e(h \upharpoonright G^{-1}\gamma\eta) \neq h \upharpoonright G^{-1}\delta\eta,$$

where we also know that for any $\varepsilon \in \eta$, $G^{-1}\gamma\varepsilon \in G^{-1}\gamma\eta$.

Proof. It is easy to check by set induction that each h_β^* and h_β are functions on the same domain as g_β^* and g_β respectively, with $\gamma \in \beta \in \alpha$ implying $h_\gamma \subseteq h_\beta$. Thus, h is a function with the same domain as g in Lemma 7.3.12, i.e. α .

Now, given any $e \in \Sigma L_\alpha$ and $\gamma, \delta \in \alpha$ where $\gamma \neq \delta$, we consider the ordinal $\beta = H^{-1}(\langle e, \langle \gamma, \delta \rangle \rangle) \in \alpha$. Then by the construction of g_β , we know that for some $\eta \in \alpha$, $G(\beta^* + \mu_\beta) = \langle \gamma, \eta \rangle$ and $G(\beta^* + \mu_\beta)^+ = \langle \delta, \eta \rangle$, whereas

$$e(h \upharpoonright \beta^* + \mu_\beta) \downarrow \rightarrow e(h \upharpoonright \beta^* + \mu_\beta) \neq h \upharpoonright (\beta^* + \mu_\beta)^+$$

holds, precisely because of our stipulation in clause (i) in the construction of h_β above. The inclusion of the sequences j_β in g_β and h_β ensures that for any $\varepsilon \in \eta$, $G^{-1}\gamma\varepsilon \in G^{-1}\gamma\eta$. $\square$

We are now in a very familiar position to repeat the work in section 5.4. Fix

$$\mathbf{l}_\alpha = \lambda f. \boldsymbol{\rho} \mathbf{p}_0 f \in \Sigma L_\alpha$$

which computes the domain of the input f , and

$$\mathbf{t}_\alpha = \lambda f x. \boldsymbol{\nu}(\boldsymbol{\rho}(\lambda t. \mathbf{i}_1(\boldsymbol{\pi} t t)(\mathbf{p}_0 t) x) f) \in \Sigma L_\alpha$$

which computes the restriction $f \upharpoonright x$. Let

$$\alpha_0 = \{\langle h \upharpoonright \beta, \check{\beta} \rangle : \beta \in \alpha\} \in V(\Sigma L_\alpha)$$

using h from Lemma 7.3.13, and we check that:

Lemma 7.3.14. $V(\Sigma L_\alpha)$ realises the following:

- (i) $\alpha_0 \in \text{Ord}$;
- (ii) $\alpha_0 \subseteq \check{\alpha}$.

Proof. Clearly

$$\lambda f. \mathbf{p}(\mathbf{l}_\alpha f)_{\iota_\tau} \Vdash \forall \beta \in \alpha_0 \beta \in \check{\alpha}.$$

For (i), we observe that for any $\langle f, \check{\beta} \rangle \in \alpha_0$ and any $\gamma \in \beta$, $\mathbf{p}(\mathbf{t}_\alpha f \gamma)_{\iota_\tau} \Vdash \check{\gamma} \in \alpha_0$. Therefore,

$$\lambda f \gamma. \mathbf{p}(\mathbf{t}_\alpha f \gamma)_{\iota_\tau} \Vdash \forall \beta \in \alpha_0 \forall \gamma \in \beta \gamma \in \alpha_0.$$

It follows from $\alpha_0 \subseteq \check{\alpha}$ that every element in α_0 is also transitive, thus $\alpha_0 \in \text{Ord}$. $\square$

Working inside $V(\Sigma L_\alpha)$, we can then construct $f_0 : \check{\alpha} \rightarrow \mathcal{P}(\check{\alpha})$ as

$$f_0(\beta) = \left\{ \gamma \in \check{\alpha} : \forall \delta \in \gamma \overline{G}^{-1}(\beta, \delta) \in \alpha_0 \right\}.$$

It is easy to check that $V(\Sigma L_\alpha)$ realises $\forall \beta \in \check{\alpha} f_0(\beta) \in \mathcal{P}(\check{\alpha}) \cap \text{Ord}$, and Lemma 7.3.13 ensures that f_0 is pairwise strongly incomparable:

Proposition 7.3.15. *For any $\gamma, \delta \in \alpha$, if some $d \Vdash f_0(\check{\gamma}) \subseteq f_0(\check{\delta})$, then $\gamma = \delta$. Namely, it follows immediately that*

$$\lambda \gamma \delta x. \iota_\tau \Vdash \forall \gamma, \delta \in \check{\alpha} (f_0(\gamma) \subseteq f_0(\delta) \rightarrow \gamma = \delta).$$

Proof. This is entirely analogous to Proposition 5.4.5, but we present it here for completeness: assume otherwise, i.e. $\gamma \neq \delta$. Reasoning inside $V(\Sigma L_\alpha)$, we know that $f_0(\check{\gamma}) \subseteq f_0(\check{\delta})$, by definition of f_0 , is equivalent to

$$\forall \beta \in \check{\alpha} \left(\forall \varepsilon \in \beta \overline{G}^{-1}(\check{\gamma}, \varepsilon) \in \alpha_0 \rightarrow \forall \varepsilon \in \beta \overline{G}^{-1}(\check{\delta}, \varepsilon) \in \alpha_0 \right),$$

so suppose that this is realised by some $d \in \Sigma L_\alpha$. Pedantically, we take the formula $\overline{G}^{-1}(\check{\gamma}, \check{\varepsilon}) \in \alpha_0$ as a shorthand for

$$\exists t \in \overline{G} \exists \eta \in \alpha_0 t = \langle \eta, \langle \check{\gamma}, \check{\varepsilon} \rangle \rangle,$$

thus if some $e \Vdash \overline{G}^{-1}(\check{\gamma}, \check{\varepsilon}) \in \alpha_0$, then $\mathbf{p}_{10}e = h \upharpoonright G^{-1}\gamma\varepsilon$ using the fact that $\overline{G}$ in $V(\Sigma L_\alpha)$ computes precisely the same map as G in the base theory.

Conversely, we can also find some function $q \in \Sigma L_\alpha$ such that, for any $\langle f, \check{\beta} \rangle \in \alpha_0$ where $G^{-1}\gamma\varepsilon \subseteq \mathbf{l}_\alpha f$,

$$q\gamma\varepsilon(\mathbf{t}_\alpha f(G^{-1}\gamma\varepsilon)) \Vdash \overline{G}^{-1}(\check{\gamma}, \check{\varepsilon}) \in \alpha_0,$$

where the precise computation performed by q is cumbersome to write but entirely routine.

Consequently, for any $e \in \Sigma L_\alpha$, take $\eta \in \alpha$ as given by Lemma 7.3.13 and $f = h \upharpoonright G^{-1}\gamma\eta$, then

$$\lambda\varepsilon. q\gamma\varepsilon(\mathbf{t}_\alpha f(G^{-1}\gamma\varepsilon)) \Vdash \forall \varepsilon \in \check{\chi} \overline{G}^{-1}(\check{\gamma}, \varepsilon) \in \alpha_0$$

where $\chi = \eta^+ = \nu(\pi\eta(\pi\eta\eta))$. If we pass this to the realiser d at the beginning of this proof, i.e. we take $p = \lambda\eta f. d(\nu(\pi\eta(\pi\eta\eta)))(\lambda\varepsilon. q\gamma\varepsilon(\mathbf{t}_\alpha f(G^{-1}\gamma\varepsilon)))\eta$, then we have $p\eta f \Vdash \overline{G}^{-1}(\check{\delta}, \check{\eta}) \in \alpha_0$ and $\mathbf{p}_{10}(p\eta f) = h \upharpoonright G^{-1}\delta\eta$.

Observe that η can be recovered as $\mathbf{p}_1(G(\mathbf{l}_\alpha f))$, thus we have a single-parameter function $e^* = \lambda f. \mathbf{p}_{10}(p(\mathbf{p}_1(G(\mathbf{l}_\alpha f))))f$, such that

$$e^*(h \upharpoonright G^{-1}\gamma\eta) \downarrow \wedge e^*(h \upharpoonright G^{-1}\gamma\eta) = h \upharpoonright G^{-1}\delta\eta,$$

which works for any $e \in \Sigma L_\alpha$ and its corresponding η . However, simply take $e = e^*$ and we have a contradiction to Lemma 7.3.13. Therefore, we must have $\gamma = \delta$ instead. $\square$

It follows immediately that:

Theorem 7.3.16 (CZF⁻ + REM). *For any ordinal β , if there exists an admissible ordinal α such that $\beta \subseteq \alpha$, then there exists a (weak) pca $\mathcal{A}$ such that $\beta \subseteq \mathcal{A}$ and*

$$V(\mathcal{A}) \models \mathcal{P}(\check{\beta}) \subseteq L.$$

Proof. The case where $\beta \subseteq \omega$ can be easily solved in $V(\text{Kl})$ by adapting the similar proof in section 5.4. Here, we will only consider $\beta > \omega$ and thus only admissible $\alpha > \omega$.

Take $\mathcal{A} = \Sigma L_\alpha$ and work inside $V(\mathcal{A})$. Note that the argument in the proof of Corollary 7.1.3 would imply that $\alpha_0 \in L$, and since $\overline{G} \in L$ by Corollary 7.3.11, we have $f_0 \in L$. Since f_0 is pairwise strongly incomparable, by Corollary 7.1.3, we have $\mathcal{P}(\check{\alpha}) \subseteq L$, and $\mathcal{P}(\check{\beta}) \subseteq L$ follows from the trivial realisability of $\check{\beta} \subseteq \check{\alpha}$. $\square$

Additionally, if the base theory satisfies exponentiation, subset collection or powerset (which have no difference under the assumption REM), then $V(\mathcal{A})$ by Lemma 6.3.1 also satisfies powerset, and we have $\mathcal{P}(\check{\beta}) \in L$. So here the obstacles with set recursive functions are no longer present.

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